

RELATIVE TOR FUNCTORS WITH RESPECT TO A SEMIDUALIZING MODULE

MARYAM SALIMI, SEAN SATHER-WAGSTAFF, ELHAM TAVASOLI,
AND SIAMAK YASSEMI

ABSTRACT. We consider relative Tor functors built from resolutions described by a semidualizing module C over a commutative noetherian ring R . We show that the bifunctors $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(-, -)$ and $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(-, -)$, defined using flat-like and projective-like resolutions, are isomorphic. We show how the vanishing of these functors characterizes the finiteness of the homological dimension \mathcal{F}_C -pd, and we use this to give a relation between the \mathcal{F}_C -pd of a given module and that of a pure submodule. On the other hand, we show that other relations that one may expect to hold similarly, fail in general. In fact, such relations force the semidualizing modules under consideration to be trivial.

INTRODUCTION

For the purposes of this paper, relative homological algebra is the study of non-traditional resolutions and the (co)homology theories (i.e., relative derived functors) that they define. By “non-traditional” we mean that these resolutions are not given directly by projective, injective, or flat modules, as they are in “absolute” homological algebra. This idea goes back to Butler and Horrocks [4] and Eilenberg and Moore [5]. This area has seen a lot of activity recently thanks to Enochs and Jenda [6] and Avramov and Martsinkovsky [2].

Much of the recent work on the derived functors that arise in this context has focused on cohomology, i.e., relative Ext; see, e.g., [2, 13, 15]. The point of this paper is to begin a pointed discussion of the properties of relative Tor. The relative homology functors that arise in this context come from resolutions that model projective resolutions and flat resolutions. Specifically, we consider proper \mathcal{P}_C -resolutions and proper \mathcal{F}_C -resolutions where C is a semidualizing module over a commutative noetherian ring R . (See Section 1 for terminology, notation, and foundational results.)

Section 2 consists of basic results about these resolutions. By their nature, these resolutions have some similar properties, but also some different properties; For instance, Proposition 2.4 shows that proper \mathcal{P}_C -resolutions behave well with respect to flat ring extensions, but the behavior of proper \mathcal{F}_C -resolutions in this context is not clear. On the other hand, restriction of scalars is well-behaved for

Date: January 30, 2012.

2010 Mathematics Subject Classification. 13D02, 13D05, 13D07.

Key words and phrases. Proper resolutions, relative homology, semidualizing modules.

This material is based on work supported by North Dakota EPSCoR and National Science Foundation Grant EPS-0814442. The research of Siamak Yassemi was in part supported by a grant from IPM (No. 91130214).

proper \mathcal{F}_C -resolutions, but not necessarily for proper \mathcal{P}_C -resolutions, as we show in Proposition 2.5.

We have four flavors of relative homology in this context. For instance, given a proper \mathcal{P}_C -resolution L of an R -module M , we have $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) = H_i(L \otimes_R N)$ for each R -module N and each integer i . The module $\mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_C}(M, N)$ is defined using a proper \mathcal{P}_C -resolution of N , and similarly, $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$ and $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N)$ are defined using proper \mathcal{F}_C -resolutions; see Definition 3.1.

Certain relations between these are obvious. For instance, commutativity of tensor product implies that $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_C}(N, M)$ and $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(N, M)$. Other relations are not obvious. For instance, it is well-known that $\mathrm{Tor}_i^R(M, N)$ can be computed using a projective resolution of M or a flat resolution of M . The corresponding result for relative Tor is our first main theorem, stated next. It is contained in Theorem 3.6.

Theorem A. *Let C be a semidualizing R -module, and let M and N be R -modules. For each i , there is a natural isomorphism $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$.*

This result allows for a certain amount of flexibility for proving results about relative Tor, as in the absolute case. This is the subject of the rest of Section 3. For instance, when M and N are finitely generated, it is straightforward to show that $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N)$ is finitely generated, while it is not obvious at all that $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$ is finitely generated. On the other hand, $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$ is well-behaved with respect to flat base change, and we get to conclude that $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N)$ is similarly well-behaved. See Propositions 3.8 and 3.10. This section concludes with relative versions of Hom-tensor adjointness, tensor evaluation, and Hom evaluation in Propositions 3.14–3.16.

Given these nice properties, one may be surprised to know that many properties of absolute Tor do not pass to the relative setting. These differences are the subject of Section 4. For instance, in Example 4.1 we show that in general we have

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) &\not\cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N) \\ \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(N, M) &\not\cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) \\ \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) &\not\cong \mathrm{Tor}_i^R(M, N). \end{aligned}$$

The remainder of this section focuses on two questions. First, Propositions 4.2–4.5 and Examples 4.6–4.7 provide classes of modules M, N such that the above “non-isomorphisms” are isomorphisms. Second, starting with Theorem 4.8, we show that the only way that the above “non-isomorphisms” are always isomorphisms is in the trivial case. For instance, here is Theorem 4.8.

Theorem B. *Assume that (R, \mathfrak{m}, k) is local, and let B and C be semidualizing R -modules. The following conditions are equivalent:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N)$ for all $i \geq 0$ and for all R -modules M, N .
- (ii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B, k) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B, k)$ for $i = 0$ and some $i > 0$.
- (iii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(k, C) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C)$ for $i = 0$ and some $i > 0$.
- (iv) $B \cong R \cong C$.

Section 5 discusses \mathcal{F}_C -pd, the homological dimension obtained from bounded proper \mathcal{F}_C -resolutions, and its relation to relative Tor. First, in Proposition 5.2 we note that this is the same homological dimension as the one calculated from

bounded acyclic \mathcal{F}_C -resolutions. From this, we deduce some flat base change results for \mathcal{F}_C -pd. In Theorems 5.6 and 5.7 we prove the next result which characterizes modules of finite \mathcal{F}_C -pd in terms of vanishing of relative Tor.

Theorem C. *Let C be a semidualizing R -module, and let M be an R -module. Given an integer $n \geq 0$, consider the following conditions:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, -) = 0$ for all $i > n$;
- (ii) $\mathrm{Tor}_{n+1}^{\mathcal{F}_C \mathcal{M}}(M, -) = 0$; and
- (iii) $\mathcal{F}_C\text{-pd}_R(M) \leq n$.
- (iv) $\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, R/\mathfrak{m}) = 0$ for all $i > n$ and for each $\mathfrak{m} \in \mathrm{m-Spec}(R)$;
- (v) $\mathrm{Tor}_{n+1}^{\mathcal{F}_C \mathcal{M}}(M, R/\mathfrak{m}) = 0$ for each $\mathfrak{m} \in \mathrm{m-Spec}(R)$; and
- (vi) $\mathcal{P}_C\text{-pd}_R(M) \leq n$.

The conditions (i)–(iii) are always equivalent. If M is finitely generated, then conditions (i)–(vi) are equivalent.

Section 6 contains the following application to pure submodules, motivated by a result of Holm and White [10]. See Theorem 6.5.

Theorem D. *Let C be a semidualizing R -module, and let $M' \subseteq M$ be a pure submodule. Then one has*

$$\mathcal{F}_C\text{-pd}_R(M) \geq \sup\{\mathcal{F}_C\text{-pd}_R(M'), \mathcal{F}_C\text{-pd}_R(M/M') - 1\}.$$

1. BACKGROUND MATERIAL

Convention 1.1. Throughout this paper R and S are commutative noetherian rings, and $\mathcal{M}(R)$ is the category of R -modules. We use the term “subcategory of $\mathcal{M}(R)$ ” to mean a “full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all R -modules M and N , if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$.” Write $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$ for the subcategories of projective, flat and injective R -modules, respectively. Write $\mathrm{m-Spec}(R)$ for the set of maximal ideals of R .

General Notions.

Definition 1.2. An R -complex is a sequence of R -module homomorphisms

$$Y = \cdots \xrightarrow{\partial_{n+1}^Y} Y_n \xrightarrow{\partial_n^Y} Y_{n-1} \xrightarrow{\partial_{n-1}^Y} \cdots$$

such that $\partial_{n-1}^Y \partial_n^Y = 0$ for each integer n . When Y is an R -complex, set $H_n(Y) = \mathrm{Ker}(\partial_n^Y) / \mathrm{Im}(\partial_{n+1}^Y)$ for each n . Given a subcategory \mathcal{X} of $\mathcal{M}(R)$, an R -complex Y is $\mathrm{Hom}_R(\mathcal{X}, -)$ -exact if the complex $\mathrm{Hom}_R(X, Y)$ is exact for each X in \mathcal{X} . The term $\mathrm{Hom}_R(-, \mathcal{X})$ -exact is defined similarly.

Given two R -complexes Y and Z , a *chain map* $f: Y \rightarrow Z$ is a sequence of R -module homomorphisms $\{f_i: Y_i \rightarrow Z_i\}$ making the obvious “ladder-diagram” commute. A chain map $f: Y \rightarrow Z$ is a *quasiisomorphism* if the induced map $H_i(f): H_i(Y) \rightarrow H_i(Z)$ is an isomorphism for each i . In general, the complexes Y and Z are *quasiisomorphic* provided that there is a sequence of quasiisomorphisms $Y \leftarrow Y^1 \rightarrow Y^2 \leftarrow \cdots \leftarrow Y^m \rightarrow Z$ for some integer m .

In this paper, resolutions are built from precovers, and coresolutions are built from preenvelopes, defined next. For more details about precovers and preenvelopes, the reader may consult [6, Chapters 5 and 6].

Definition 1.3. Let \mathcal{X} be a subcategory of $\mathcal{M}(R)$ and let M be an R -module. An \mathcal{X} -precover of M is an R -module homomorphism $\varphi: X \rightarrow M$, where $X \in \mathcal{X}$, and such that the sequence

$$\mathrm{Hom}_R(X', \varphi): \mathrm{Hom}_R(X', X) \rightarrow \mathrm{Hom}_R(X', M) \rightarrow 0$$

is exact for every $X' \in \mathcal{X}$. If every R -module admits \mathcal{X} -precover, then the class \mathcal{X} is *precovering*. The terms \mathcal{X} -preenvelope and *preenveloping* are defined dually.

Assume that \mathcal{X} is precovering. Then each R -module M has an *augmented proper \mathcal{X} -resolution*, that is, an R -complex

$$X^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\tau} M \longrightarrow 0$$

such that $\mathrm{Hom}_R(Y, X^+)$ is exact for all $Y \in \mathcal{X}$. The truncated complex

$$X = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow 0$$

is a *proper \mathcal{X} -resolution* of M . The \mathcal{X} -projective dimension of M is

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is a proper } \mathcal{X}\text{-resolution of } M\}.$$

Proper \mathcal{X} -coresolutions and \mathcal{X} -id are defined dually.

When \mathcal{X} is the class of projective R -modules, we write $\mathrm{pd}_R(M)$ for the associated homological dimension and call it the *projective dimension* of M . Similarly, the flat and injective dimensions of M are denoted $\mathrm{fd}_R(M)$ and $\mathrm{id}_R(M)$.

Remark 1.4. Let \mathcal{X} be a precovering subcategory of $\mathcal{M}(R)$. We note explicitly that augmented proper \mathcal{X} -resolutions need not be exact.

According to our definitions, we have $\mathcal{X}\text{-pd}_R(0) = -\infty$. The modules of \mathcal{X} -projective dimension zero are the non-zero modules in \mathcal{X} .

Note that projective resolutions (in the usual sense) are automatically proper. Also, note that augmented proper flat resolutions are automatically exact.

The following result shows that there is some versatility in proper flat resolutions. It is for use in Proposition 5.2.

Lemma 1.5. *Let N be a module such that there is an exact sequence*

$$0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow N \rightarrow 0$$

where each G_i is flat. Let F be a proper flat resolution of N , and set $K_n = \mathrm{Im}(\partial_{n+1}^F)$. Then the truncation

$$\tilde{F}^+ = (0 \rightarrow K_n \rightarrow F_{n-1} \xrightarrow{\partial_{n-1}^F} \cdots \xrightarrow{\partial_1^F} F_0 \rightarrow N \rightarrow 0)$$

is also a proper flat resolution of N .

Proof. Note that Remark 1.4 implies that F^+ is exact, so \tilde{F}^+ is also exact. A standard version of Schanuel's Lemma implies that K_n is flat. Let G be a flat R -module. We need to show that $\mathrm{Hom}_R(G, \tilde{F}^+)$ is exact. The left exactness of $\mathrm{Hom}_R(G, -)$ shows that $\mathrm{Hom}_R(G, \tilde{F}^+)$ is exact in degrees $\geq n-1$. The fact that F is proper provides the exactness in degrees $< n-1$. \square

Remark 1.6. The difference between flat resolutions (in the usual sense) and proper flat resolutions is subtle. For instance, every R -module has a proper flat resolution since $\mathcal{F}(R)$ is precovering by [3]. However, some flat resolutions are proper, and others are not. Moreover, the next example shows that even bounded

flat resolutions need not be proper. On the other hand, Lemma 1.5 shows that the classical flat dimension of N is the same as $\text{fd}_R(N)$, that is, the homological dimension defined using flat resolutions (in the usual sense) is the same as the homological dimension defined using proper flat resolutions. See also Proposition 5.2. Of course, these subtleties do not come up for pd and id since projective resolutions and injective coresolutions are automatically proper.

Example 1.7. Assume that (R, \mathfrak{m}, k) is a local, non-complete, Gorenstein domain such that $\dim(R) = 1$. For instance, we can take $R = \mathbb{Z}_{(p)}$ or $k[X]_{(X)}$ where k is a field. The augmented minimal injective resolution of R (over itself) has the form

$$X = (0 \rightarrow R \rightarrow Q \xrightarrow{\alpha} E \rightarrow 0)$$

where $Q = Q(R)$ is the field of fractions of R and $E = E_R(k)$ is the injective hull of k . This is also an augmented flat resolution of E , in the usual sense. To show that this flat resolution is not proper, we show that $\text{Hom}_R(\widehat{R}, X)$ is not exact

$$\text{Hom}_R(\widehat{R}, X) = (0 \rightarrow \text{Hom}_R(\widehat{R}, R) \rightarrow \text{Hom}_R(\widehat{R}, Q) \xrightarrow{\text{Hom}_R(\widehat{R}, \alpha)} \text{Hom}_R(\widehat{R}, E) \rightarrow 0)$$

where \widehat{R} is the \mathfrak{m} -adic completion of R . (This suffices since \widehat{R} is flat over R .) In fact, the right-most homology module in this complex is

$$\text{Coker}(\text{Hom}_R(\widehat{R}, \alpha)) = \text{Ext}_R^1(\widehat{R}, R)$$

which is non-zero by [8, Main Theorem 2.5]. See also [1, Propositions 4.2 and 4.5] for specific computations of $\text{Ext}_R^1(\widehat{R}, R)$.

Semidualizing Modules and Relative Homological Algebra.

Semidualizing modules, defined next, form the basis for our categories of interest. These objects go back at least to Vasconcelos [16], but were rediscovered by others.

Definition 1.8. A finitely generated R -module C is *semidualizing* if the natural “homothety morphism” $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}_R^i(C, C) = 0$ for $i \geq 1$. An R -module D is *dualizing* if it is semidualizing and has finite injective dimension.

Let C be a semidualizing R -module. We set

$\mathcal{P}_C(R)$ = the subcategory of modules $M \cong P \otimes_R C$ for some $P \in \mathcal{P}(R)$

$\mathcal{F}_C(R)$ = the subcategory of modules $M \cong F \otimes_R C$ for some $F \in \mathcal{F}(R)$

$\mathcal{I}_C(R)$ = the subcategory of modules $M \cong \text{Hom}_R(C, I)$ for some $I \in \mathcal{I}(R)$.

The R -modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ are called *C-projective*, *C-flat* and *C-injective*, respectively.

Remark 1.9. Let C be a semidualizing R -module. In [10] Holm and White prove that the classes $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are closed under coproducts and summands and the class $\mathcal{I}_C(R)$ is closed under products and summands. Also, they proved that the classes $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are precovering, and the class $\mathcal{I}_C(R)$ is preenveloping. Since R is noetherian and C is finitely generated, it is straightforward to show that the class $\mathcal{F}_C(R)$ is closed under products, and $\mathcal{I}_C(R)$ is closed under coproducts.

Remark 1.10. Let C be a semidualizing R -module. Then C is cyclic if and only if it is free, if and only if $C \cong R$. Similarly, $\text{pd}_R(C) < \infty$ if and only if C is projective (necessarily of rank 1). If R is Gorenstein and local, then $C \cong R$. If $R \rightarrow S$ is a flat ring homomorphism, then $S \otimes_R C$ is a semidualizing S -module. In the local

setting, these facts are discussed in [12, Section 1]. For the non-local case, see [11, Chapter 2].

The next classes were also introduced by Vasconcelos [16].

Definition 1.11. Let C be a semidualizing R -module. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R -modules M such that:

- (i) $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$ for all $i \geq 1$, and
- (ii) the natural map $M \rightarrow \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R -modules M such that:

- (i) $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$ for all $i \geq 1$, and
- (ii) the natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, M) \xrightarrow{\xi_M^C} M$ is an isomorphism.

Remark 1.12. Let C be a semidualizing R -module. The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ satisfy the “two-of-three property”: given an exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ of R -module homomorphisms, if two of the M_i are in $\mathcal{A}_C(R)$ or in $\mathcal{B}_C(R)$, then so is the third M_i ; see [10, Corollary 6.3].

The class $\mathcal{A}_C(R)$ contains all R -modules of finite flat dimension and all modules of finite \mathcal{I}_C -injective dimension. The Bass class $\mathcal{B}_C(R)$ contains all R -modules of finite injective dimension and all modules M such that there is an exact sequence

$$0 \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0$$

such that each $L_i \in \mathcal{F}_C(R)$; hence all modules of finite \mathcal{P}_C -projective dimension. (See [10, Corollary 6.1] and [15, 1.9]).¹ See also Proposition 5.2.

Foxby equivalence [15, Theorem 2.8] states the following:

- (a) An R -module M is in $\mathcal{B}_C(R)$ if and only if $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C(R)$.
- (b) An R -module M is in $\mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(R)$.

The Auslander and Bass classes for $C = R$ are trivial: $\mathcal{B}_R(R) = \mathcal{M}(R) = \mathcal{A}_R(R)$.

The next two results are for use in the proofs of Propositions 4.3 and 5.2.

Lemma 1.13. *Let C be a semidualizing R -module, and let X, Y be R -complexes such that $X_i, Y_i \in \mathcal{A}_C(R)$ for each index i . Assume that X and Y are both either bounded above or bounded below.*

- (a) *If X is exact, then so is $C \otimes_R X$.*
- (b) *If $f: X \rightarrow Y$ is a quasiisomorphism, then so is $C \otimes_R f: C \otimes_R X \rightarrow C \otimes_R Y$.*
- (c) *If X and Y are quasiisomorphic and bounded below, then so are $C \otimes_R X$ and $C \otimes_R Y$.*

Proof. (a) The result holds if X is a short exact sequence, since $\mathrm{Tor}_1^R(C, X_i) = 0$ for each i . The general result follows by breaking X into short exact sequences. Note that this uses the two-of-three property for $\mathcal{A}_C(R)$ from Remark 1.12.

(b) This follows by applying part (a) to the mapping cone of f .

¹Note that there seems to be a bit of ambiguity in [10, Corollary 6.1]. Before [10, 1.3] the authors state that all resolutions are defined by precovers. In [10, 1.3], the authors define proper resolutions in terms of precovers, but in [10, 1.4] they define \mathcal{X} -pd in terms of \mathcal{X} -resolutions, with no mention of properness. Then in [10, Corollary 6.1], the authors are clearly assuming that their bounded augmented resolutions are exact. For \mathcal{P}_C -pd and \mathcal{I}_C -id, this is covered in [15, Corollary 2.10], which we recall in Fact 1.15. However, \mathcal{F}_C -pd is not covered there, at least not explicitly. We take care of this in Proposition 5.2.

(c) This follows from part (b), since there are quasiisomorphisms $f: P \rightarrow X$ and $g: P \rightarrow Y$ for some bounded below complex P of projective R -modules. Note that this uses the fact that every projective R -module is in $\mathcal{A}_C(R)$; see Remark 1.12. \square

Similarly, we have the following.

Lemma 1.14. *Let C be a semidualizing R -module, and let X, Y be R -complexes such that $X_i, Y_i \in \mathcal{B}_C(R)$ for each index i . Assume that X and Y are both either bounded above or bounded below.*

- (a) *If X is exact, then so is $\mathrm{Hom}_R(C, X)$.*
- (b) *If $f: X \rightarrow Y$ is a quasiisomorphism, then so is $\mathrm{Hom}_R(C, f): \mathrm{Hom}_R(C, X) \rightarrow \mathrm{Hom}_R(C, Y)$.*
- (c) *If X and Y are quasiisomorphic and bounded above, then so are $\mathrm{Hom}_R(C, X)$ and $\mathrm{Hom}_R(C, Y)$.*

Next, we recall some results from [15, Corollary 2.10 and Theorem 2.11]. It compares directly to Proposition 5.2.

Fact 1.15. Let C be a semidualizing R -module, and let M be an R -module.

- (a) One has $\mathcal{P}_C\text{-pd}_R(M) \leq n$ if and only if there is an exact sequence

$$0 \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0$$

such that each $L_i \in \mathcal{P}_C(R)$.

- (b) One has $\mathcal{I}_C\text{-id}_R(M) \leq n$ if and only if there is an exact sequence

$$0 \rightarrow M \rightarrow J^0 \rightarrow \cdots \rightarrow J^n \rightarrow 0$$

such that each $J^i \in \mathcal{I}_C(R)$.

- (c) $\mathcal{P}_C\text{-pd}_R(M) = \mathrm{pd}_R(\mathrm{Hom}_R(C, M))$.
- (d) $\mathcal{I}_C\text{-id}_R(M) = \mathrm{id}_R(C \otimes_R M)$.
- (e) $\mathcal{P}_C\text{-pd}_R(C \otimes_R M) = \mathrm{pd}_R(M)$.
- (f) $\mathcal{I}_C\text{-id}_R(\mathrm{Hom}_R(C, M)) = \mathrm{id}_R(M)$.

The following functors are studied in [13, 15]. We work with them in Propositions 3.14–3.16, and use them in the proof of Theorem 5.6.

Definition 1.16. Let C be a semidualizing R -module, and let M and N be R -modules. Let L be a proper \mathcal{P}_C -resolution of M , and let J be a proper \mathcal{I}_C -coresolution of N . For each i , set

$$\begin{aligned} \mathrm{Ext}_{\mathcal{P}_C\mathcal{M}}^i(M, N) &:= \mathrm{H}_{-i}(\mathrm{Hom}_R(L, N)) \\ \mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^i(M, N) &:= \mathrm{H}_{-i}(\mathrm{Hom}_R(M, J)). \end{aligned}$$

Fact 1.17. Let C be a semidualizing R -module, and let M be an R -module. Given an integer $n \geq 0$, we know from [15, Theorem 3.2(b)] that the following conditions are equivalent:

- (i) $\mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^i(-, M) = 0$ for all $i > n$;
- (ii) $\mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^{n+1}(-, M) = 0$; and
- (iii) $\mathcal{I}_C\text{-id}_R(M) \leq n$.

Relations Between Semidualizing Modules.

Over a local ring, the “isomorphism” relation on the class of semidualizing modules is pretty good at distinguishing between semidualizing modules with different properties. For instance, if B and C are semidualizing modules over a local ring, then $\mathcal{B}_C(R) = \mathcal{B}_B(R)$ if and only if $C \cong B$; see [7] for this result and other similar results. On the other hand, when R is not local, one has to work a bit harder to distinguish between homologically similar semidualizing modules. The following discussion is also from [7].

Definition 1.18. Let $\text{Pic}(R)$ denote the Picard group of R . The elements of $\text{Pic}(R)$ are the isomorphism classes $[P]$ of finitely generated rank 1 projective R -modules P , that is, the finitely generated projective R -modules P such that $P_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ for all maximal (equivalently, for all prime) ideals $\mathfrak{m} \subset R$. The group structure on $\text{Pic}(R)$ is given by tensor product $[P][Q] = [P \otimes_R Q]$, and the identity in $\text{Pic}(R)$ is $[R]$. Inverses are given by duality $[P]^{-1} = [\text{Hom}_R(P, R)]$, and similarly for division: $[P]^{-1}[Q] = [\text{Hom}_R(P, Q)]$.

Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes $[C]$ of semidualizing R -modules.

Fact 1.19. Let M be an R -module. Then M is a finitely generated rank 1 projective R -module if and only if M is a semidualizing R -module of finite projective dimension, by [7, Remark 4.7]. So we have $\text{Pic}(R) \subseteq \mathfrak{S}_0(R)$. Also, there is an action of $\text{Pic}(R)$ on $\mathfrak{S}_0(R)$ given by $[P][C] = [P \otimes_R C]$.

Definition 1.20. The equivalence relation defined by the action of $\text{Pic}(R)$ on $\mathfrak{S}_0(R)$ is denoted \approx : given $[B], [C] \in \mathfrak{S}_0(R)$ we have $[B] \approx [C]$ provided that $[B]$ and $[C]$ are in the same orbit under $\text{Pic}(R)$, that is, provided that there is an element $[P] \in \text{Pic}(R)$ such that $C \cong P \otimes_R B$. Write $B \approx C$ when $[B] \approx [C]$.

Fact 1.21. Given semidualizing R -modules B and C , the following conditions are equivalent:

- (i) $B \approx C$.
- (ii) $B_{\mathfrak{m}} \cong C_{\mathfrak{m}}$ for all maximal (equivalently, for all prime) ideals $\mathfrak{m} \subset R$.
- (iii) $\mathcal{B}_B(R) = \mathcal{B}_C(R)$.
- (iv) $\mathcal{A}_B(R) = \mathcal{A}_C(R)$.
- (v) $B \in \mathcal{B}_C(R)$ and $C \in \mathcal{B}_B(R)$.

See [7, Theorems 1.4, Propositions 5.1 and 5.4].

Lemma 1.22. Let B and C be semidualizing R -modules such that $B \approx C$, and let $[P] \in \text{Pic}(R)$ such that $C \cong P \otimes_R B$. Then one has $\mathcal{P}_B(R) = \mathcal{P}_C(R)$ and $\mathcal{F}_B(R) = \mathcal{F}_C(R)$ and $\mathcal{I}_B(R) = \mathcal{I}_C(R)$.

Proof. Let Q be a projective R -module. The assumption $C \cong P \otimes_R B$ implies that

$$C \otimes_R Q \cong (P \otimes_R B) \otimes_R Q \cong B \otimes_R (P \otimes_R Q).$$

Since $P \otimes_R Q$ is projective, this implies that $\mathcal{P}_C(R) \subseteq \mathcal{P}_B(R)$. The reverse containment is proved similarly, using the isomorphism $B \cong \text{Hom}_R(P, R) \otimes_R C$. The equalities $\mathcal{F}_B(R) = \mathcal{F}_C(R)$ and $\mathcal{I}_B(R) = \mathcal{I}_C(R)$ are verified similarly. \square

Two Lemmas on Semidualizing Modules.

The next two results are for use in Section 4.

Lemma 1.23. Assume that (R, \mathfrak{m}, k) is local, and let C be a semidualizing R -module. Consider the following conditions:

- (i) $C \cong R$.
- (ii) $C \otimes_R C$ is free.
- (iii) $\text{pd}_R(C \otimes_R C) < \infty$.

Then one has (i) \iff (ii) \implies (iii). If R is artinian, then the conditions (i)–(iii) are equivalent.

Proof. The implications (i) \implies (ii) \implies (iii) are straightforward. When R is artinian, the implication (iii) \implies (ii) follows from the Auslander-Buchsbaum formula.

(ii) \implies (i) Assume that $C \otimes_R C$ is free, and let $\beta = \beta_0(C)$ denote the minimal number of generators of C . By Nakayama's Lemma, the module $C \otimes_R C$ is minimally generated by β^2 many elements, so we have $C \otimes_R C \cong R^{\beta^2}$. On the other hand, the surjection $R^\beta \twoheadrightarrow C$ gives a surjection

$$C^\beta \twoheadrightarrow C \otimes_R C \cong R^{\beta^2}$$

by right exactness of tensor product. This splits, so R^{β^2} is a direct summand of C^β . Taking endomorphism rings, we conclude that $\text{End}(R^{\beta^2}) \cong R^{\beta^4}$ is a direct summand of $\text{End}_R(C^\beta) \cong R^{\beta^2}$. In particular, this implies that $\beta^4 \leq \beta^2$, which implies that $\beta = 1$. It follows that C is cyclic, so $C \cong R$ by Remark 1.10. \square

For perspective, the ring R in the next result is isomorphic to the “trivial extension” or “idealization” $k \ltimes k^2$.

Lemma 1.24. *Let k be a field, and set $R = k[X, Y]/(X, Y)^2$. If C is a non-free semidualizing R -module, then C is dualizing for R and $C \otimes_R C \cong k^4$.*

Proof. As C is not free, it is non-cyclic by Remark 1.10, so we have $\beta := \beta_0(C) \geq 2$. The ring R is local with maximal ideal $\mathfrak{m} = (X, Y)R$ such that $\mathfrak{m}^2 = 0$.

We first show that C is dualizing for R . Since C is non-free, and $\mathfrak{m}^2 = 0$, it follows that there is an exact sequence

$$0 \rightarrow k^a \rightarrow R^\beta \rightarrow C \rightarrow 0$$

with $a \neq 0$. The conditions $\text{Ext}_R^i(R, C) = 0 = \text{Ext}_R^i(C, C)$ for all $i \geq 1$ imply that $\text{Ext}_R^i(k^a, C) = 0$ for all $i \geq 1$. We have $a \neq 0$, so $\text{Ext}_R^i(k, C) = 0$ for all $i \geq 1$. Thus C has finite injective dimension, and C is dualizing by definition.

The structure of the dualizing module for this ring is pretty well understood. For instance, we have $\beta = \mu_R^0(R) = 2$. Moreover, we can describe C in terms of generators and relations, as follows. The multi-graded structure on R is represented in the following diagram:



where each bullet represents the corresponding monomial in R . It follows that $C \cong E_R(k) \cong k \cdot X^{-1} \oplus k \cdot Y^{-1} \oplus k \cdot 1$ with multi-graded module structure given by the formulas

$$\begin{array}{lll} X \cdot 1 = 0 & X \cdot X^{-1} = 1 & X \cdot Y^{-1} = 0 \\ Y \cdot 1 = 0 & Y \cdot Y^{-1} = 1 & Y \cdot X^{-1} = 0. \end{array}$$

In other words, the multi-graded structure is represented by the following diagram:



where each bullet represents the corresponding monomial in C . Using this grading, one can show that $C \supseteq RY^{-1} \cong R/XR$ and $C/RX^{-1} \cong k$. In particular, there is an exact sequence

$$0 \rightarrow R/XR \rightarrow C \rightarrow k \rightarrow 0. \quad (1.24.1)$$

Also, we see that $XC = k \cdot 1$, so $C/XC \cong k^2$.

We claim that $4 \leq \text{len}_R(C \otimes_R C) \leq 6$. To check this, consider the exact sequence

$$0 \rightarrow k \rightarrow C \rightarrow k^2 \rightarrow 0$$

coming from the equalities $\beta = 2$, $\text{len}_R(C) = 3$, and $\mathfrak{m}^2 = 0$. The right exactness of $C \otimes_R -$ implies that the next sequence is exact:

$$C \otimes_R k \rightarrow C \otimes_R C \rightarrow C \otimes_R k^2 \rightarrow 0.$$

Since $C \otimes_R k \cong k^\beta = k^2$, it follows that

$$4 \leq \text{len}_R(C \otimes_R C) \leq 4 + 2 = 6$$

as claimed.

Next, we show that $\text{len}_R(C \otimes_R C) \leq 4$. For this, we apply $C \otimes_R -$ to the sequence (1.24.1) to obtain the next exact sequence

$$C/XC \rightarrow C \otimes_R C \rightarrow C \otimes_R k \rightarrow 0.$$

As we noted above, we have $C/XC \cong k^2 \cong C \otimes_R k$, so additivity of length implies that $\text{len}_R(C \otimes_R C) \leq 4$.

It follows that $\text{len}_R(C \otimes_R C) = 4$. Also, we have $\beta_0(C \otimes_R C) = \beta_0(C)^2 = \beta^2 = 4$, by Nakayama's Lemma. That is, the modules $C \otimes_R C$ and $(C \otimes_R C)/\mathfrak{m}(C \otimes_R C)$ both have length 4. Since $(C \otimes_R C)/\mathfrak{m}(C \otimes_R C)$ is a homomorphic image of $C \otimes_R C$, it follows that $C \otimes_R C \cong (C \otimes_R C)/\mathfrak{m}(C \otimes_R C) \cong k^4$ as desired. \square

2. PROPER RESOLUTIONS

Throughout this section, C is a semidualizing R -module, and M is an R -module.

The results of this section document some properties of proper \mathcal{F}_C -resolutions and proper \mathcal{P}_C -resolutions. We begin with some notation.

Construction 2.1. Let F be a flat (e.g., projective) resolution of $\text{Hom}_R(C, M)$.

$$F^+ = \cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \xrightarrow{\tau} \text{Hom}_R(C, M) \rightarrow 0.$$

Let $\xi_M^C: C \otimes_R \text{Hom}_R(C, M) \rightarrow M$ denote the natural evaluation map, and let $(C \otimes_R F)^\pm$ denote the following complex

$$(C \otimes_R F)^\pm = \cdots \xrightarrow{C \otimes \partial_2^F} C \otimes_R F_1 \xrightarrow{C \otimes \partial_1^F} C \otimes_R F_0 \xrightarrow{\xi_M^C \circ (C \otimes \tau)} M \rightarrow 0$$

where $C \otimes_R \text{Hom}_R(C, M) \xrightarrow{\xi_M^C} M$ is the natural evaluation map. In other words, $(C \otimes_R F)^\pm$ is obtained by augmenting the complex $C \otimes_R F$ by the composition

$$C \otimes_R F_0 \xrightarrow{C \otimes \tau} C \otimes_R \text{Hom}_R(C, M) \xrightarrow{\xi_M^C} M.$$

The next lemma is implicit in [15].

- Lemma 2.2.** (a) *If F is a proper flat resolution of $\text{Hom}_R(C, M)$, then $C \otimes_R F$ is a proper \mathcal{F}_C -resolution of M .*
 (b) *If G is a proper \mathcal{F}_C -resolution of M , then $\text{Hom}_R(C, G)$ is a proper flat resolution of $\text{Hom}_R(C, M)$.*
 (c) *If P is a projective resolution of $\text{Hom}_R(C, M)$, then $C \otimes_R P$ is a proper \mathcal{P}_C -resolution of M .*
 (d) *If Q is a proper \mathcal{P}_C -resolution of M , then $\text{Hom}_R(C, Q)$ is a projective resolution of $\text{Hom}_R(C, M)$.*

Proof. (a) To show that $C \otimes_R F$ is a proper \mathcal{F}_C -resolution of M , it suffices to show that the complex $(C \otimes_R F)^\pm$ from Construction 2.1 is $\text{Hom}_R(\mathcal{F}_C, -)$ -exact. Let L be a flat R -module. We need to show that the complex $\text{Hom}_R(C \otimes_R L, (C \otimes_R F)^\pm)$ is exact. This complex has the following form.

$$\cdots \rightarrow \text{Hom}_R(C \otimes_R L, C \otimes_R F_1) \rightarrow \text{Hom}_R(C \otimes_R L, C \otimes_R F_0) \rightarrow \text{Hom}_R(C \otimes_R L, M) \rightarrow 0$$

By Hom-tensor adjointness, this is isomorphic to the next complex where $(-)' = \text{Hom}_R(C, C \otimes_R -)$:

$$\cdots \rightarrow \text{Hom}_R(L, F_1') \rightarrow \text{Hom}_R(L, F_0') \rightarrow \text{Hom}_R(L, \text{Hom}_R(C, M)) \rightarrow 0.$$

Since each F_i is in $\mathcal{A}_C(R)$, this is isomorphic to a complex of the following form:

$$\cdots \rightarrow \text{Hom}_R(L, F_1) \rightarrow \text{Hom}_R(L, F_0) \rightarrow \text{Hom}_R(L, \text{Hom}_R(C, M)) \rightarrow 0.$$

It is straightforward (but tedious) to show that this complex is isomorphic to $\text{Hom}_R(L, F^+)$ which is exact since F is a proper flat resolution of $\text{Hom}_R(C, M)$. Thus, $(C \otimes_R F)^\pm$ is $\text{Hom}_R(\mathcal{F}_C, -)$ -exact, as desired.

(b) For each C -flat module Y , the module $\text{Hom}_R(C, Y)$ is flat. Thus, the fact that $\text{Hom}_R(C, G)$ is a proper flat resolution of $\text{Hom}_R(C, M)$ follows as in part (a).

Parts (c) and (d) are proved similarly. \square

Lemma 2.3. *Let X be an R -complex. Then X is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact if and only if $\text{Hom}_R(C, X)$ is exact.*

Proof. The forward implication is from the condition $C \in \mathcal{P}_C(R)$. For the reverse implication, let P be a projective R -module. Since $\text{Hom}_R(C, X)$ is exact, the fact that P is projective implies that the complex $\text{Hom}_R(C \otimes_R P, X) \cong \text{Hom}_R(P, \text{Hom}_R(C, X))$ is exact, as desired. \square

The next result is the first of several applications of Lemma 2.2. We do not know whether the corresponding result for proper \mathcal{F}_C -resolutions holds. See, however, Corollary 5.3.

Proposition 2.4. *Let $R \rightarrow S$ be a flat ring homomorphism. If L is a proper \mathcal{P}_C -resolution of M over R , then $S \otimes_R L$ is a proper $\mathcal{P}_{S \otimes_R C}$ -resolution of $S \otimes_R M$ over S .*

Proof. We augment $S \otimes_R L$ in the natural way, via the given augmentation for L , so that we have $(S \otimes_R L)^+ \cong S \otimes_R (L^+)$. Thus, the notation $S \otimes_R L^+$ is unambiguous.

To show that $S \otimes_R L$ is a proper $\mathcal{P}_{S \otimes_R C}$ -resolution of $S \otimes_R M$, first note that each module in L is of the form $L_i \cong C \otimes_R P_i$ for some projective R -module P_i . Hence, the module $S \otimes_R P_i$ is projective over S . The isomorphisms

$$S \otimes_R L_i \cong S \otimes_R (C \otimes_R P_i) \cong (S \otimes_R C) \otimes_S (S \otimes_R P_i)$$

imply that $S \otimes_R L_i \in \mathcal{P}_{S \otimes_R C}(S)$.

Next, since L is a proper \mathcal{P}_C -resolution of M over R , the complex $\text{Hom}_R(C, L^+)$ is exact. The fact that S is flat over R implies that the next complex

$$\text{Hom}_S(S \otimes_R C, S \otimes_R L^+) \cong S \otimes_R \text{Hom}_R(C, L^+)$$

is also exact. Hence, Lemma 2.3 implies that $S \otimes_R L^+$ is $\text{Hom}_S(\mathcal{P}_{S \otimes_R C}, -)$ -exact, so $S \otimes_R L$ is proper, as desired. \square

The previous result works for projectives, but not necessarily for flats. On the other hand, the next result works for flats, but not for projectives.

Proposition 2.5. *Let $R \rightarrow S$ be a flat ring homomorphism. Assume that M is an S -module, and let L be a proper $\mathcal{F}_{S \otimes_R C}$ -resolution of M over S . Then L is a proper \mathcal{F}_C -resolution of M over R .*

Proof. Each module L_i is of the form $L_i \cong (S \otimes_R C) \otimes_S F_i \cong C \otimes_R F_i$ for some flat S -module F_i . Since S is flat over R , it follows that each F_i is flat over R , so each L_i is in $\mathcal{F}_C(R)$. To show that L is proper over R , let G be a flat R -module:

$$\begin{aligned} \text{Hom}_R(C \otimes_R G, L^+) &\cong \text{Hom}_R(C \otimes_R G, \text{Hom}_S(S, L^+)) \\ &\cong \text{Hom}_S(S \otimes_R (C \otimes_R G), L^+) \\ &\cong \text{Hom}_S((S \otimes_R C) \otimes_S (S \otimes_R G), L^+). \end{aligned}$$

Since G is flat over R , we know that $S \otimes_R G$ is flat over S , and it follows that $(S \otimes_R C) \otimes_S (S \otimes_R G)$ is $S \otimes_R C$ -flat over S . Thus, the fact that L is proper over S implies that the displayed complexes are exact, so L is proper over R . \square

Of course, localization gives useful examples of flat ring homomorphisms.

Corollary 2.6. *Let U be a multiplicatively closed subset of R . If L is a proper \mathcal{P}_C -resolution of M over R , then $U^{-1}L$ is a proper $\mathcal{P}_{U^{-1}C}$ -resolution of $U^{-1}M$ over $U^{-1}R$.*

Corollary 2.7. *Let U be a multiplicatively closed subset of R , and assume that M is a $U^{-1}R$ -module. If L is a proper $\mathcal{F}_{U^{-1}C}$ -resolution of M over $U^{-1}R$, then L is a proper \mathcal{F}_C -resolution of M over R .*

The proofs of parts (a) and (b) of the next result are necessarily different because $\text{Hom}_R(L, -)$ does not commute with coproducts in general.

Lemma 2.8. *Let $\{M_j\}_{j \in J}$ be a set of R -modules. For each $j \in J$, let X_j be a proper \mathcal{F}_C -resolution of M_j , and let Y_j be a proper \mathcal{P}_C -resolution of M_j .*

- (a) *The product $\prod_j X_j$ is a proper \mathcal{F}_C -resolution of $\prod_j M_j$.*
- (b) *The coproduct $\coprod_j Y_j$ is a proper \mathcal{P}_C -resolution of $\coprod_j M_j$.*

Proof. (a) Since $\mathcal{F}_C(R)$ is closed under products by Remark 1.9, the complex $\prod_j X_j$ consists of modules in $\mathcal{F}_C(R)$. The augmentation map for $(\prod_j X_j)^+$ is the natural one induced on products, so we have $(\prod_j X_j)^+ = \prod_j X_j^+$. To show that this complex is $\text{Hom}_R(\mathcal{F}_C, -)$ -exact, let $L \in \mathcal{F}_C(R)$ and compute:

$$\text{Hom}_R(L, \prod_j X_j^+) \cong \prod_i \text{Hom}_R(L, X_j^+).$$

Since each complex $\text{Hom}_R(L, X_j^+)$ is exact by assumption, the same is true of the displayed complex, as desired.

(b) As in part (a), the modules in $\coprod_j Y_j$ are C -projective. To show that $\coprod_j Y_j$ is proper, Lemma 2.3 shows that we need only check that $\text{Hom}_R(C, \coprod_j Y_j^+)$ is exact. Since C is finitely generated, we know that $\text{Hom}_R(C, -)$ commutes with coproducts, so the desired result follows as in the proof of part (a). \square

3. RELATIVE HOMOLOGY

In this section, C is a semidualizing R -module, and M and N are R -modules.

In our setting, there are four different relative Tor-modules to consider. They are gotten by resolving in the first slot by modules in $\mathcal{P}_C(R)$ or $\mathcal{F}_C(R)$, and similarly for the second slot.

Definition 3.1. Let Q be a proper \mathcal{P}_C -resolution of M , and let G be a proper \mathcal{F}_C -resolution of M . For each $i \geq 0$, set

$$\begin{aligned} \text{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N) &:= \text{H}_i(Q \otimes_R N) & \text{Tor}_i^{\mathcal{M} \mathcal{P}_C}(N, M) &:= \text{H}_i(N \otimes_R Q) \\ \text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N) &:= \text{H}_i(G \otimes_R N) & \text{Tor}_i^{\mathcal{M} \mathcal{F}_C}(N, M) &:= \text{H}_i(N \otimes_R G). \end{aligned}$$

Remark 3.2. The properness assumption on the resolutions in Definition 3.1 guarantee that these relative Tor constructions are independent of the choice of resolutions and functorial in both arguments. See [6, Section 8.2]. Also, there are natural transformations of bifunctors

$$\begin{aligned} \text{Tor}_0^{\mathcal{P}_C \mathcal{M}}(-, -) &\rightarrow - \otimes_R - & \text{Tor}_0^{\mathcal{M} \mathcal{P}_C}(-, -) &\rightarrow - \otimes_R - \\ \text{Tor}_0^{\mathcal{F}_C \mathcal{M}}(-, -) &\rightarrow - \otimes_R - & \text{Tor}_0^{\mathcal{M} \mathcal{F}_C}(-, -) &\rightarrow - \otimes_R -. \end{aligned}$$

In general, these are not isomorphisms, as we see in Example 4.1 below.

Given the symmetric nature of the definitions, one has

$$\text{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N) \cong \text{Tor}_i^{\mathcal{M} \mathcal{P}_C}(N, M) \quad \text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N) \cong \text{Tor}_i^{\mathcal{M} \mathcal{F}_C}(N, M).$$

Thus, every result for $\text{Tor}_i^{\mathcal{P}_C \mathcal{M}}(-, -)$ has a companion result for $\text{Tor}_i^{\mathcal{M} \mathcal{P}_C}(-, -)$, and similarly for $\text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(-, -)$ and $\text{Tor}_i^{\mathcal{M} \mathcal{F}_C}(-, -)$. For the sake of brevity, we do not state both versions explicitly in most cases.

Example 3.3. In the trivial case $C = R$, we have $\mathcal{F}_R(R) = \mathcal{F}(R)$ and $\mathcal{P}_R(R) = \mathcal{P}(R)$, and the relative Tors are the same as the absolute Tors.

$$\text{Tor}_i^{\mathcal{P}_R \mathcal{M}}(-, -) \cong \text{Tor}_i^{\mathcal{M} \mathcal{P}_R}(-, -) \cong \text{Tor}_i^{\mathcal{F}_R \mathcal{M}}(-, -) \cong \text{Tor}_i^{\mathcal{M} \mathcal{F}_R}(-, -) \cong \text{Tor}_i^R(-, -)$$

The following long exact sequences come from [6, Theorem 8.2.3].

Proposition 3.4. Let $\mathbb{L} = (0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0)$ be a complex of R -modules.

(a) If \mathbb{L} is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact (i.e., if $\text{Hom}_R(C, \mathbb{L})$ is exact, e.g., if $L' \in \mathcal{B}_C(R)$), then there is a long exact sequence

$$\cdots \text{Tor}_1^{\mathcal{P}_C \mathcal{M}}(L'', N) \rightarrow \text{Tor}_0^{\mathcal{P}_C \mathcal{M}}(L', N) \rightarrow \text{Tor}_0^{\mathcal{P}_C \mathcal{M}}(L, N) \rightarrow \text{Tor}_0^{\mathcal{P}_C \mathcal{M}}(L'', N) \rightarrow 0$$

that is natural in \mathbb{L} and N .

(b) If \mathbb{L} is $\text{Hom}_R(\mathcal{F}_C, -)$ -exact, then there is a long exact sequence

$$\cdots \text{Tor}_1^{\mathcal{F}_C \mathcal{M}}(L'', N) \rightarrow \text{Tor}_0^{\mathcal{F}_C \mathcal{M}}(L', N) \rightarrow \text{Tor}_0^{\mathcal{F}_C \mathcal{M}}(L, N) \rightarrow \text{Tor}_0^{\mathcal{F}_C \mathcal{M}}(L'', N) \rightarrow 0$$

that is natural in \mathbb{L} and N .

Construction 3.5. For each i , there is a natural transformation of bifunctors

$$\varrho_i: \operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(-, -) \rightarrow \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(-, -).$$

To construct ϱ_i , let Q be a proper \mathcal{P}_C -resolution of M , and let G be a proper \mathcal{F}_C -resolution of M . The containment $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$ implies that the augmented resolution G^+ is $\operatorname{Hom}_R(\mathcal{P}_C, -)$ -exact. As in the proof of the functoriality of the relative Tors, it follows that there is a morphism of complexes $Q^+ \rightarrow G^+$ that is an isomorphism in degree -1 . Furthermore, this morphism is unique up to homotopy. Thus, the induced morphism $Q \otimes_R N \rightarrow G \otimes_R N$ gives rise to the desired map by taking homology.

The next result compares to [15, Theorem 4.1] which has similar formulas for relative Ext. This contains Theorem A from the introduction.

Theorem 3.6. *For each i , there are natural isomorphisms*

$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \xrightarrow{\cong} \operatorname{Tor}_i^R(\operatorname{Hom}_R(C, M), C \otimes_R N) \xrightarrow{\cong} \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$$

and the morphism $\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(-, -) \xrightarrow{\varrho_i} \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(-, -)$ is an isomorphism.

Proof. Let F be a proper flat resolution of $\operatorname{Hom}_R(C, M)$. Lemma 2.2(a) implies that $C \otimes_R F$ is a proper \mathcal{F}_C -resolution of M , so we have

$$\begin{aligned} \operatorname{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) &\cong \operatorname{H}_i((C \otimes_R F) \otimes_R N) \\ &\cong \operatorname{H}_i(F \otimes_R (C \otimes_R N)) \\ &\cong \operatorname{Tor}_i^R(\operatorname{Hom}_R(C, M), C \otimes_R N). \end{aligned}$$

The naturality of this isomorphism comes from the naturality of the constructions, and similarly for $\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N)$.

Let $P \rightarrow F$ be a lift of the identity map on $\operatorname{Hom}_R(C, M)$. Then the induced map $(C \otimes_R P)^\pm \rightarrow (C \otimes_R F)^\pm$ is of the form $Q^+ \rightarrow G^+$, as in Construction 3.5. It follows that $\varrho_i(M, N)$ is the map gotten by taking homology in the map

$$(C \otimes_R P) \otimes_R N \rightarrow (C \otimes_R F) \otimes_R N.$$

Of course, this is equivalent to taking homology in the map

$$P \otimes_R (C \otimes_R N) \rightarrow F \otimes_R (C \otimes_R N).$$

The fact that $\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, M), C \otimes_R N)$ can be computed using P or F implies that the induced maps on homology are isomorphisms, as desired. \square

The assumptions on \mathbb{L} in the next result are satisfied, e.g., when \mathbb{L} is exact and $L'' \in \mathcal{A}_C(R)$.

Corollary 3.7. *Let $\mathbb{L} = (0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0)$ be a complex of R -modules such that $C \otimes_R \mathbb{L}$ is exact. Then there are long exact sequences*

$$\begin{aligned} \cdots \operatorname{Tor}_1^{\mathcal{P}_C\mathcal{M}}(N, L'') &\rightarrow \operatorname{Tor}_0^{\mathcal{P}_C\mathcal{M}}(N, L') \rightarrow \operatorname{Tor}_0^{\mathcal{P}_C\mathcal{M}}(N, L) \rightarrow \operatorname{Tor}_0^{\mathcal{P}_C\mathcal{M}}(N, L'') \rightarrow 0 \\ \cdots \operatorname{Tor}_1^{\mathcal{F}_C\mathcal{M}}(N, L'') &\rightarrow \operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(N, L') \rightarrow \operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(N, L) \rightarrow \operatorname{Tor}_0^{\mathcal{F}_C\mathcal{M}}(N, L'') \rightarrow 0 \end{aligned}$$

that are natural in \mathbb{L} and N .

Proof. Apply $C \otimes_R -$ to get the exact sequence

$$0 \rightarrow C \otimes_R L' \rightarrow C \otimes_R L \rightarrow C \otimes_R L'' \rightarrow 0.$$

Now take the long exact sequence in $\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, N), -)$ using Theorem 3.6. \square

Theorem 3.6 allows for a certain amount of flexibility for relative Tor, in the same way that flat and projective resolutions give flexibility for absolute Tor. For instance, in the next result, it is not clear that a finitely generated module M has a proper \mathcal{F}_C -resolution L such that each L_i is finitely generated.

Proposition 3.8. *Assume that M and N are finitely generated over R . For all i the modules $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N)$ and $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$ are finitely generated over R .*

Proof. Since C , M and N are finitely generated, so are $\mathrm{Hom}_R(C, M)$ and $C \otimes_R N$, and hence so is $\mathrm{Tor}_i^R(\mathrm{Hom}_R(C, M), C \otimes_R N)$. Thus, the desired conclusion follows from Theorem 3.6.

Alternately, given a degreewise finite R -free resolution F of $\mathrm{Hom}_R(C, M)$, the complex $C \otimes_R F$ is a degreewise finite proper \mathcal{P}_C -resolution of M by Lemma 2.2(c). It follows that the complex $(C \otimes_R F) \otimes_R N$ is degreewise finite, so the homology modules $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N)$ are finitely generated over R . \square

Proposition 3.9. *Let $\{N_j\}_{j \in J}$ be a set of R -modules.*

(a) *For each i , there are isomorphisms*

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, \coprod_j N_j) &\cong \coprod_j \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N_j) \\ \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(\coprod_j N_j, M) &\cong \coprod_j \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(N_j, M) \end{aligned}$$

and similarly for $\mathrm{Tor}^{\mathcal{F}_C\mathcal{M}}$.

(b) *If M is finitely generated, then for each i , there are isomorphisms*

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, \prod_j N_j) &\cong \prod_j \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N_j) \\ \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(\prod_j N_j, M) &\cong \prod_j \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(N_j, M) \end{aligned}$$

and similarly for $\mathrm{Tor}^{\mathcal{F}_C\mathcal{M}}$.

Proof. (a) For the first isomorphism, let X be a proper \mathcal{P}_C -resolution of M , and use the isomorphism $X \otimes_R \coprod_j N_j \cong \coprod_j X \otimes_R N_j$. For the second isomorphism, use Lemma 2.8(b). The isomorphisms for $\mathrm{Tor}^{\mathcal{F}_C\mathcal{M}}$ follow using Theorem 3.6.

(b) If M is finitely generated, then $-\otimes_R M$ commutes with arbitrary products. Hence, the isomorphism $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(\prod_j N_j, M) \cong \prod_j \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(N_j, M)$ follows from Lemma 2.8(a), and the corresponding isomorphisms for $\mathrm{Tor}^{\mathcal{P}_C\mathcal{M}}$ follow using Theorem 3.6. Finally, as in the proof of Proposition 3.8, the module M has a proper \mathcal{P}_C -resolution X such that each X_i is finitely generated. Hence, the functor $X \otimes_R -$ respects arbitrary products, and the final isomorphisms follow. \square

Next, we discuss flat base change.

Proposition 3.10. *Let $R \rightarrow S$ be a flat ring homomorphism. Then for all i there are S -module isomorphisms*

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_{S \otimes_R C}\mathcal{M}}(S \otimes_R M, S \otimes_R N) &\cong S \otimes_R \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \\ \mathrm{Tor}_i^{\mathcal{F}_{S \otimes_R C}\mathcal{M}}(S \otimes_R M, S \otimes_R N) &\cong S \otimes_R \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N). \end{aligned}$$

Proof. In view of Theorem 3.6, it suffices to justify the first isomorphism. Let L be a proper \mathcal{P}_C -resolution of M . Proposition 2.4 implies that $S \otimes_R L$ is a proper

$\mathcal{P}_{S \otimes_R C}$ -resolution of $S \otimes_R M$. This explains the first step in the next sequence

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_{S \otimes_R C} \mathcal{M}}(S \otimes_R M, S \otimes_R N) &\cong \mathrm{H}_i((S \otimes_R L) \otimes_S (S \otimes_R N)) \\ &\cong \mathrm{H}_i(S \otimes_R (L \otimes_R N)) \\ &\cong S \otimes_R \mathrm{H}_i(L \otimes_R N) \\ &\cong S \otimes_R \mathrm{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N). \end{aligned}$$

The third isomorphism is by the R -flatness of S . \square

Of course, localization is a special case of flat base change:

Corollary 3.11. *Let U be a multiplicatively closed subset of R . Then for all i there are $U^{-1}R$ -module isomorphisms*

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_{U^{-1}C} \mathcal{M}}(U^{-1}M, U^{-1}N) &\cong U^{-1} \mathrm{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N) \\ \mathrm{Tor}_i^{\mathcal{F}_{U^{-1}C} \mathcal{M}}(U^{-1}M, U^{-1}N) &\cong U^{-1} \mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N). \end{aligned}$$

Proposition 3.12. *Let $R \rightarrow S$ be a flat ring homomorphism, and assume that N is an S -module. Then for all i there are S -module isomorphisms*

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_{S \otimes_R C} \mathcal{M}}(S \otimes_R M, N) &\cong \mathrm{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N) \\ \mathrm{Tor}_i^{\mathcal{F}_{S \otimes_R C} \mathcal{M}}(S \otimes_R M, N) &\cong \mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N). \end{aligned}$$

Proof. As in the proof of Proposition 3.10, the first isomorphism in the next sequence is from Proposition 2.4

$$\mathrm{Tor}_i^{\mathcal{P}_{S \otimes_R C} \mathcal{M}}(S \otimes_R M, N) \cong \mathrm{H}_i((S \otimes_R L) \otimes_S N) \cong \mathrm{H}_i(L \otimes_R N) \cong \mathrm{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N).$$

This is the first of our desired isomorphisms; the second one follows by 3.6. \square

Proposition 3.13. *Let $R \rightarrow S$ be a flat ring homomorphism, and assume that M is an S -module. Then for all i there are S -module isomorphisms*

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_{S \otimes_R C} \mathcal{M}}(S \otimes_R M, N) &\cong \mathrm{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N) \\ \mathrm{Tor}_i^{\mathcal{F}_{S \otimes_R C} \mathcal{M}}(S \otimes_R M, N) &\cong \mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N). \end{aligned}$$

Proof. Let L be a proper $\mathcal{F}_{S \otimes_R C}$ -resolution of M over S . Proposition 2.5 shows that L is a proper \mathcal{F}_C -resolution of M over R . The desired isomorphisms now follow as in the proof of Proposition 3.12. \square

The next three results provide relative versions of some standard results for absolute homology, beginning with Hom-tensor adjointness.

Proposition 3.14. *Let I be an injective R -module. For all $i \geq 0$ one has*

$$\mathrm{Ext}_{\mathcal{P}_C \mathcal{M}}^i(M, \mathrm{Hom}_R(N, I)) \cong \mathrm{Hom}_R(\mathrm{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N), I) \quad (3.14.1)$$

$$\mathrm{Ext}_{\mathcal{M} \mathcal{T}_C}^i(M, \mathrm{Hom}_R(N, I)) \cong \mathrm{Hom}_R(\mathrm{Tor}_i^{\mathcal{M} \mathcal{P}_C}(M, N), I). \quad (3.14.2)$$

Proof. The first isomorphism in the next sequence follows from [15, Theorem 4.1]

$$\begin{aligned} \mathrm{Ext}_{\mathcal{P}_C \mathcal{M}}^i(M, \mathrm{Hom}_R(N, I)) &\cong \mathrm{Ext}_R^i(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(C, \mathrm{Hom}_R(N, I))) \\ &\cong \mathrm{Ext}_R^i(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(C \otimes_R N, I)) \\ &\cong \mathrm{Hom}_R(\mathrm{Tor}_i^R(\mathrm{Hom}_R(C, M), C \otimes_R N), I) \\ &\cong \mathrm{Hom}_R(\mathrm{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N), I). \end{aligned}$$

The second isomorphism is by Hom-tensor adjointness, and the remaining steps follow from [6, Theorem 3.2.1] and Theorem 3.6. This explains (3.14.1), and (3.14.2) is established similarly. \square

The next result is a version of tensor evaluation for relative Ext.

Proposition 3.15. *Assume that M is finitely generated over R , and let F be a flat R -module. For all $i \geq 0$ there are isomorphisms*

$$\mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^i(M, N) \otimes_R F \cong \mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^i(M, N \otimes_R F) \quad (3.15.1)$$

$$\mathrm{Ext}_{\mathcal{P}_C\mathcal{M}}^i(M, N) \otimes_R F \cong \mathrm{Ext}_{\mathcal{P}_C\mathcal{M}}^i(M, N \otimes_R F). \quad (3.15.2)$$

Proof. The isomorphism (3.15.1) follows from the next display

$$\begin{aligned} \mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^i(M, N) \otimes_R F &\cong \mathrm{Ext}_R^i(C \otimes_R M, C \otimes_R N) \otimes_R F \\ &\cong \mathrm{Ext}_R^i(C \otimes_R M, (C \otimes_R N) \otimes_R F) \\ &\cong \mathrm{Ext}_R^i(C \otimes_R M, C \otimes_R (N \otimes_R F)) \\ &\cong \mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^i(M, N \otimes_R F) \end{aligned}$$

which is from [15, Theorem 4.1] and [6, Theorem 3.2.15]. The isomorphism (3.15.2) is established similarly. \square

Next, we have a version of Hom-evaluation for the relative setting.

Proposition 3.16. *Assume that M is finitely generated over R , and let I be an injective R -module. Then for all $i \geq 0$ there are isomorphisms*

$$\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, \mathrm{Hom}_R(N, I)) \cong \mathrm{Hom}_R(\mathrm{Ext}_{\mathcal{P}_C\mathcal{M}}^i(M, N), I) \quad (3.16.1)$$

$$\mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_C}(M, \mathrm{Hom}_R(N, I)) \cong \mathrm{Hom}_R(\mathrm{Ext}_{\mathcal{M}\mathcal{I}_C}^i(M, N), I). \quad (3.16.2)$$

Proof. The first isomorphism in the next display is from Theorem 3.6:

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, \mathrm{Hom}_R(N, I)) &\cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(C, M), C \otimes_R \mathrm{Hom}_R(N, I)) \\ &\cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(\mathrm{Hom}_R(C, N), I)) \\ &\cong \mathrm{Hom}_R(\mathrm{Ext}_R^i(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(C, N)), I) \\ &\cong \mathrm{Hom}_R(\mathrm{Ext}_{\mathcal{P}_C\mathcal{M}}^i(M, N), I). \end{aligned}$$

The second and third isomorphisms are from [6, Theorem 3.2.11 and 3.2.13], and the fourth isomorphism follows from [15, Theorem 4.1]. This explains (3.16.1), and (3.16.2) is established similarly. \square

4. COMPARISON OF RELATIVE HOMOLOGIES

In this section, B, C are semidualizing R -modules, and M, N are R -modules.

Using Theorem 3.6, we show that relative Tors do not satisfy the naive version of balance, that they are not commutative, and that they do not agree with absolute Tor in general.

Example 4.1. Assume that (R, \mathfrak{m}, k) is local and that C is not free, that is, that C is not cyclic. We show that

$$\begin{aligned}\mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(k, C) &\not\cong \mathrm{Tor}_i^{\mathcal{M}^{\mathcal{F}^C}}(k, C) \\ \mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(C, k) &\not\cong \mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(k, C) \\ \mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(k, C) &\not\cong \mathrm{Tor}_i^R(k, C)\end{aligned}$$

for all i , at least in a specific example.

Let $\beta = \beta_0^R(C) \geq 2$. It is straightforward to show that $\mathrm{Tor}_i^{\mathcal{M}^{\mathcal{F}^C}}(k, C) = 0$ for all $i \geq 1$ and that $\mathrm{Tor}_0^{\mathcal{M}^{\mathcal{F}^C}}(k, C) \cong k \otimes_R C \cong k^\beta$; see also Proposition 4.2 and Theorem 5.6. From Theorem 3.6, we have

$$\mathrm{Tor}_0^{\mathcal{F}^C\mathcal{M}}(k, C) \cong \mathrm{Hom}_R(C, k) \otimes_R (C \otimes_R C) \cong k^\beta \otimes_R (C \otimes_R C) \cong k^{\beta^3}.$$

This is not isomorphic to

$$\mathrm{Tor}_0^R(C, k) \cong \mathrm{Tor}_0^{\mathcal{F}^C\mathcal{M}}(C, k) \cong k^\beta \cong \mathrm{Tor}_0^{\mathcal{M}^{\mathcal{F}^C}}(k, C)$$

as $\beta \geq 2$, so $\mathrm{Tor}_0^{\mathcal{F}^C\mathcal{M}}(C, k) \not\cong \mathrm{Tor}_0^{\mathcal{F}^C\mathcal{M}}(k, C) \not\cong \mathrm{Tor}_0^{\mathcal{M}^{\mathcal{F}^C}}(k, C)$ and $\mathrm{Tor}_0^{\mathcal{F}^C\mathcal{M}}(k, C) \not\cong \mathrm{Tor}_0^R(k, C)$.

Again using Theorem 3.6, for $i \geq 1$ we have

$$\mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(k, C) \cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(C, k), C \otimes_R C) \cong \mathrm{Tor}_i^R(k, C \otimes_R C)^\beta \quad (4.1.1)$$

and

$$\mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(C, k) = 0 = \mathrm{Tor}_i^{\mathcal{M}^{\mathcal{F}^C}}(k, C).$$

Thus, to show that $\mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(C, k) \not\cong \mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(k, C) \not\cong \mathrm{Tor}_i^{\mathcal{M}^{\mathcal{F}^C}}(k, C)$ in general, it suffices to find an example such that $\mathrm{Tor}_i^R(k, C \otimes_R C) \neq 0$ for all $i \geq 1$, that is, such that $\mathrm{pd}_R(C \otimes_R C) = \infty$.² This is supplied by Lemma 1.23, assuming that R is artinian.

Finally, we give a specific example where $\mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(k, C) \not\cong \mathrm{Tor}_i^R(k, C)$ for all i . Note that (4.1.1) shows that $\mathrm{Tor}_i^{\mathcal{F}^C\mathcal{M}}(k, C) \cong k^{\beta \cdot \beta_i(C \otimes_R C)}$. Since $\mathrm{Tor}_i^R(k, C) \cong k^{\beta_i(C)}$, it suffices to provide an example where

$$\beta \cdot \beta_i(C \otimes_R C) > \beta_i(C) \quad (4.1.2)$$

for all $i \geq 1$.

Set $R = k[X, Y]/(X, Y)^2$, so we have $\mathfrak{m}^2 = 0$. Let $C = \mathrm{Hom}_k(R, k)$, which is dualizing for R and has $\beta = \mu_R^0(R) = 2$. Lemma 1.24 implies that $C \otimes_R C \cong k^4$, so we have

$$\beta_i(C \otimes_R C) = 4\beta_i(k) = 4 \cdot 2^i = 2^{i+2} \quad (4.1.3)$$

for all $i \geq 0$. Also, we have $\mathrm{len}_R(C) = 3$. Since $\mathfrak{m}^2 = 0$, it follows that there is an exact sequence

$$0 \rightarrow k^3 \rightarrow R^2 \rightarrow C \rightarrow 0 \quad (4.1.4)$$

which implies that $\beta_1(C) = 3$. Dimension shifting in the sequence (4.1.4) implies that

$$\beta_i(C) = 3\beta_{i-1}(k) = 3 \cdot 2^{i-1}$$

for all $i \geq 1$. From this, one easily deduces the inequality (4.1.2) for all $i \geq 1$, using (4.1.3) with the equality $\beta = 2$.

²We believe that this is true in general, under the assumption that C is not free; see [7].

In general, Example 4.1 shows that many naive properties fail for relative homology. We continue this section by giving some special cases where these naive properties do hold.

Proposition 4.2. *If the natural map $C \otimes_R \text{Hom}_R(C, M) \rightarrow M$ is an isomorphism (e.g., if $M \in \mathcal{B}_C(R)$), then $\text{Tor}_0^{\mathcal{F}_C \mathcal{M}}(M, -) \cong M \otimes_R -$.*

Proof. Again, by Theorem 3.6 we have

$$\begin{aligned} \text{Tor}_0^{\mathcal{F}_C \mathcal{M}}(M, N) &\cong \text{Hom}_R(C, M) \otimes_R (C \otimes_R N) \\ &\cong (\text{Hom}_R(C, M) \otimes_R C) \otimes_R N \\ &\cong M \otimes_R N \end{aligned}$$

where the last isomorphism is from the assumption $C \otimes_R \text{Hom}_R(C, M) \cong M$. \square

In general, we have $\text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, -) \not\cong \text{Tor}_i^R(M, -)$ by Example 4.1, even when $M \in \mathcal{B}_C(R)$. The next result gives conditions on M and N guaranteeing that the isomorphism $\text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N) \cong \text{Tor}_i^R(M, N)$ does hold.

Proposition 4.3. *If $M \in \mathcal{B}_C(R)$ and $N \in \mathcal{A}_C(R)$, then for each i there are isomorphisms*

$$\begin{aligned} \text{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N) &\cong \text{Tor}_i^R(M, N) \\ \text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N) &\cong \text{Tor}_i^R(M, N). \end{aligned}$$

Proof. Let P be a projective resolution of $\text{Hom}_R(C, M)$, and let Q be a projective resolution of N . Lemma 2.2(c) implies that $C \otimes_R P$ is a proper \mathcal{P}_C -resolution of M .

We use the tensor product of complexes. Since Q is a bounded below complex of projective R -modules, it respects quasiisomorphisms. This explains the second isomorphism in the next sequence:

$$\begin{aligned} \text{Tor}_i^R(M, N) &\cong H_i(M \otimes_R Q) \\ &\cong H_i((C \otimes_R P) \otimes_R Q) \\ &\cong H_i((C \otimes_R P) \otimes_R N) \\ &\cong \text{Tor}_i^{\mathcal{P}_C \mathcal{M}}(M, N). \end{aligned}$$

The first isomorphism is from the balance of Tor. The fourth isomorphism is by definition. It remains to explain the third isomorphism.

Since Q is a projective resolution of N , there is a quasiisomorphism $Q \xrightarrow{\sim} N$. Since P is a bounded below complex of projective R -modules, the functor $P \otimes_R -$ respects quasiisomorphisms. So there is a quasiisomorphism $P \otimes_R Q \xrightarrow{\sim} P \otimes_R N$. Lemma 1.13(b) implies that the induced map $C \otimes_R P \otimes_R Q \xrightarrow{\sim} C \otimes_R P \otimes_R N$ is also a quasiisomorphism. By the associativity of tensor product, this implies that the complexes $(C \otimes_R P) \otimes_R Q$ and $(C \otimes_R P) \otimes_R N$ are quasiisomorphic; in particular, they have isomorphic homologies, as desired. \square

The best results (as best we know) for balance and commutativity are the following.

Proposition 4.4. *If $M \in \mathcal{B}_B(R) \cap \mathcal{A}_C(R)$ and $N \in \mathcal{B}_C(R) \cap \mathcal{A}_B(R)$, then one has $\text{Tor}_i^{\mathcal{F}_B \mathcal{M}}(M, N) \cong \text{Tor}_i^{\mathcal{M} \mathcal{F}_C}(M, N)$ for all $i \geq 0$.*

Proof. Proposition 4.3 implies that

$$\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N)$$

for all $i \geq 0$. \square

The next result is proved similarly.

Proposition 4.5. *If $M, N \in \mathcal{B}_C(R) \cap \mathcal{A}_C(R)$, then one has $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(N, M)$ for all $i \geq 0$.*

We next give examples of some modules that satisfy the hypotheses of the previous two results. First, we show how to find some modules in $\mathcal{B}_C(R) \cap \mathcal{A}_B(R)$ and $\mathcal{B}_B(R) \cap \mathcal{A}_C(R)$.

Example 4.6. By [7, Corollary 3.8], the following conditions are equivalent:

- (i) $C \in \mathcal{A}_B(R)$.
- (ii) $B \in \mathcal{A}_C(R)$, and
- (iii) $\mathrm{Tor}_i^R(B, C) = 0$ for all $i \geq 1$ and $B \otimes_R C$ is a semidualizing R -module.

For instance, if A is a semidualizing R -module such that $A \in \mathcal{B}_C(R)$, then $B = \mathrm{Hom}_R(C, A)$ satisfies these conditions with $B \otimes_R C \cong A$.

Assume that the above conditions are satisfied. Then $B \in \mathcal{B}_B(R) \cap \mathcal{A}_C(R)$, and it follows that $\mathcal{F}_B(R) \subseteq \mathcal{B}_B(R) \cap \mathcal{A}_C(R)$. By the two-of-three property from Remark 1.12, every module of finite \mathcal{F}_B -projective dimension is in $\mathcal{B}_B(R) \cap \mathcal{A}_C(R)$. Similarly, every module of finite \mathcal{F}_C -projective dimension is in $\mathcal{B}_C(R) \cap \mathcal{A}_B(R)$.

Another class of modules like this is from [13, Fact 3.13].³ Assume that R is Cohen-Macaulay with a dualizing module D . Then $D \in \mathcal{B}_C(R)$, so the dual $C^\dagger := \mathrm{Hom}_R(C, D)$ is a semidualizing R -module such that $C \in \mathcal{A}_{C^\dagger}(R)$. Every module of finite $\mathcal{G}(\mathcal{P}_C)$ -projective dimension is in $\mathcal{B}_C(R) \cap \mathcal{A}_{C^\dagger}(R)$, and every module of finite $\mathcal{G}(\mathcal{P}_{C^\dagger})$ -projective dimension is in $\mathcal{B}_{C^\dagger}(R) \cap \mathcal{A}_C(R)$ by symmetry since $C \cong C^{\dagger\dagger}$. Also, every module of finite $\mathcal{G}(\mathcal{I}_{C^\dagger})$ -injective dimension is in $\mathcal{B}_C(R) \cap \mathcal{A}_{C^\dagger}(R)$, and every module of finite $\mathcal{G}(\mathcal{I}_C)$ -injective dimension is in $\mathcal{B}_{C^\dagger}(R) \cap \mathcal{A}_C(R)$.

Finding modules that are in $\mathcal{A}_C(R) \cap \mathcal{B}_C(R)$ is more difficult in general.

Example 4.7. Assume that R is a domain. Then the quotient field $Q(R)$ is both flat and injective, so it is in $\mathcal{A}_C(R) \cap \mathcal{B}_C(R)$ for each semidualizing R -module C .

Of course, if $B \cong R \cong C$, then we have $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(M, N) \cong \mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N)$; see Example 3.3. The following result shows that, in the local case, this is the only way to achieve balance of all M and N ; it is Theorem B from the introduction. We discuss the non-local case below because it requires more technology.

Theorem 4.8. *Assume that (R, \mathfrak{m}, k) is local. The following are equivalent:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(X, Y) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X, Y)$ for all $i \geq 0$ and for all R -modules X, Y .
- (ii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B, k) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B, k)$ for $i = 0$ and some $i \geq 1$.
- (iii) $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(k, C) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C)$ for $i = 0$ and some $i \geq 1$.
- (iv) $B \cong R \cong C$.

³Since this example is only given to put our results in perspective, we refer the reader to [13] for the relevant notations and definitions.

Proof. We verify the implications (i) \implies (ii) \implies (iv) \implies (i). The implications (i) \implies (iii) \implies (iv) \implies (i) are verified similarly. Of course, the implication (i) \implies (ii) is trivial, and the implication (iv) \implies (i) is from Example 3.3.

(ii) \implies (iv) We exploit Theorem 3.6:

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B, k) &\cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(B, B), B \otimes_R k) \\ &\cong \mathrm{Tor}_i^R(R, k^{\beta_0(B)}) \\ &\cong \begin{cases} k^{\beta_0(B)} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases} \\ \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B, k) &\cong \mathrm{Tor}_i^R(C \otimes_R B, \mathrm{Hom}_R(C, k)) \\ &\cong \mathrm{Tor}_i^R(C \otimes_R B, k^{\beta_0(C)}) \\ &\cong k^{\beta_i(C \otimes_R B)\beta_0(C)}. \end{aligned}$$

Assuming that $\mathrm{Tor}_0^{\mathcal{F}_B\mathcal{M}}(B, k) \cong \mathrm{Tor}_0^{\mathcal{M}\mathcal{F}_C}(B, k)$, we conclude that

$$\beta_0(B) = \beta_0(C \otimes_R B)\beta_0(C) = \beta_0(B)\beta_0(C)^2.$$

Since $\beta_0(B) \neq 0$, it follows that $\beta_0(C) = 1$. So C is cyclic, and therefore $C \cong R$ by Remark 1.10. Assuming $\mathrm{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B, k) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(B, k)$ for some $i \geq 1$, we have

$$0 = \beta_i(C \otimes_R B)\beta_0(C) = \beta_i(B).$$

It follows that $\mathrm{pd}_R(B) < \infty$, so Remark 1.10 implies that $B \cong R$, as desired. \square

Here is a similar result for commutativity.

Corollary 4.9. *Assume that (R, \mathfrak{m}, k) is local. The following are equivalent:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(X, Y) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(Y, X)$ for all $i \geq 0$ and for all R -modules X, Y .
- (ii) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X, Y) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(Y, X)$ for all $i \geq 0$ and for all R -modules X, Y .
- (iii) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, k) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C)$ for $i = 0$ and some $i \geq 1$.
- (iv) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(C, k) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C)$ for $i = 0$ and some $i \geq 1$.
- (v) $C \cong R$.

Proof. This follows from Theorem 4.8 with $B = C$. For instance, we verify the implication (iii) \implies (v). Assume that $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, k) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C)$ for $i = 0$ and some $i \geq 1$. Then Remark 3.2 implies that

$$\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, k) \cong \mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, C) \cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(C, k)$$

for $i = 0$ and some $i \geq 1$, so we have $C \cong R$ by Theorem 4.8(ii) \implies (iv). \square

Here is another result of the same flavor.

Corollary 4.10. *Assume that (R, \mathfrak{m}, k) is local. The following are equivalent:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(X, Y) \cong \mathrm{Tor}_i^R(X, Y)$ for all $i \geq 0$ and for all R -modules X, Y .
- (ii) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(X, Y) \cong \mathrm{Tor}_i^R(X, Y)$ for all $i \geq 0$ and for all R -modules X, Y .
- (iii) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(C, k) \cong \mathrm{Tor}_i^R(C, k)$ for some $i \geq 1$.
- (iv) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, C) \cong \mathrm{Tor}_i^R(k, C)$ for some $i \geq 1$.
- (v) $C \cong R$.
- (vi) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, k) \cong \mathrm{Tor}_i^R(k, k)$ for some $i \geq 0$.
- (vii) $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(k, k) \cong \mathrm{Tor}_i^R(k, k)$ for some $i \geq 0$.

Proof. (iv) \implies (v) Example 3.3 implies that $\text{Tor}_i^R(-, -) \cong \text{Tor}_i^{\mathcal{F}_R\mathcal{M}}(-, -)$. Thus, we can apply Theorem 4.8 with $B = R$. Indeed, Proposition 4.2 implies that $\text{Tor}_0^{\mathcal{M}^{\mathcal{F}_C}}(k, C) \cong \text{Tor}_0^R(k, C) \cong \text{Tor}_0^{\mathcal{M}^{\mathcal{F}_R}}(k, C)$. Condition (iv) translates to say that $\text{Tor}_i^{\mathcal{M}^{\mathcal{F}_C}}(k, C) \cong \text{Tor}_i^R(k, C) \cong \text{Tor}_i^{\mathcal{M}^{\mathcal{F}_R}}(k, C)$ for some $i \geq 1$, so we have $C \cong R$ by Theorem 4.8(iii) \implies (iv).

The equivalence of conditions (i)–(v) follows similarly from Theorem 4.8 with $B = R$. And Example 3.3 justifies the implications (v) \implies (vi) and (v) \implies (vii).

(vi) \implies (v) Since $\text{Hom}_R(C, k) \cong k^{\beta_0(C)} \cong C \otimes_R k$, we have

$$\text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, k) \cong \text{Tor}_i^R(\text{Hom}_R(C, k), C \otimes_R k) \cong \text{Tor}_i^R(k, k)^{\beta_0(C)^2} \cong k^{\beta_0(C)^2\beta_i(k)}$$

and of course $\text{Tor}_i^R(k, k) \cong k^{\beta_i(k)}$. Suppose that $\text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, k) \cong \text{Tor}_i^R(k, k)$ for some $i \geq 0$. It follows that $\beta_i(k) = \beta_0(C)^2\beta_i(k)$, so either $\beta_0(C) = 1$ or $\beta_i(k) = 0$. In the first case, we have $C \cong R$ as before. In the other case, the ring R is regular, hence Gorenstein, so $C \cong R$ by Remark 1.10.

The implication (vii) \implies (v) is verified similarly. \square

Remark 4.11. Note that Theorem 4.8 and Corollary 4.9 do not contain versions of the conditions (vi) and (vii) of Corollary 4.10. Indeed, for Corollary 4.9 this is because we always have

$$\text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(k, k) \cong k^{\beta_0(C)^2\beta_i(k)} \cong \text{Tor}_i^{\mathcal{M}^{\mathcal{F}_C}}(k, k)$$

as the proof of Corollary 4.9 shows. Similarly, if one assumes in Theorem 4.8 that $\text{Tor}_i^{\mathcal{F}_B\mathcal{M}}(k, k) \cong \text{Tor}_i^{\mathcal{M}^{\mathcal{F}_C}}(k, k)$, then the only conclusion one would be able to draw from this is that $\beta_0(B) = \beta_0(C)$, which is not enough to guarantee that B and C are isomorphic, let alone isomorphic to R .

Now we prove the non-local versions of the results 4.8–4.10. First, we have the following. Recall the relation \approx from Definition 1.20.

Proposition 4.12. *Assume that $B \approx C$, and let $[P] \in \text{Pic}(R)$ such that $C \cong P \otimes_R B$. For each i , there are natural isomorphisms*

$$\begin{aligned} \text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) &\cong \text{Tor}_i^{\mathcal{F}_B\mathcal{M}}(M, N) \\ \text{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) &\cong \text{Tor}_i^{\mathcal{P}_B\mathcal{M}}(M, N) \end{aligned}$$

Proof. This follows immediately from Lemma 1.22. \square

Corollary 4.13. *Let $[C] \in \text{Pic}(R)$. For each i there are isomorphisms*

$$\begin{aligned} \text{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) &\cong \text{Tor}_i^R(M, N) \\ \text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, N) &\cong \text{Tor}_i^R(M, N). \end{aligned}$$

Proof. The condition $[C] \in \text{Pic}(R)$ is equivalent to $C \approx R$. So, the result follows from Example 3.3 and Proposition 4.12. \square

Corollary 4.14. *The following conditions are equivalent:*

- (i) $\text{Tor}_i^{\mathcal{F}_B\mathcal{M}}(X, Y) \cong \text{Tor}_i^{\mathcal{M}^{\mathcal{F}_C}}(X, Y)$ for all $i \geq 0$ and for all R -modules X, Y .
- (ii) $\text{Tor}_i^{\mathcal{F}_B\mathcal{M}}(B, R/\mathfrak{m}) \cong \text{Tor}_i^{\mathcal{M}^{\mathcal{F}_C}}(B, R/\mathfrak{m})$ for $i = 0$, for some $i \geq 1$, and for all $\mathfrak{m} \in \text{m-Spec}(R)$.
- (iii) $\text{Tor}_i^{\mathcal{F}_B\mathcal{M}}(R/\mathfrak{m}, C) \cong \text{Tor}_i^{\mathcal{M}^{\mathcal{F}_C}}(R/\mathfrak{m}, C)$ for $i = 0$, for some $i \geq 1$, and for all $\mathfrak{m} \in \text{m-Spec}(R)$.
- (iv) $B \approx R \approx C$, i.e., $[B], [C] \in \text{Pic}(R)$.

Proof. As in the proof of Theorem 4.8, we verify the implications (ii) \implies (iv) \implies (i). The implication (iv) \implies (i) is from Corollary 4.13.

(ii) \implies (iv) Assume that $\mathrm{Tor}_i^{\mathcal{F}_B \mathcal{M}}(R/\mathfrak{m}, C) \cong \mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(R/\mathfrak{m}, C)$ for all $i \geq 0$, for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)$. Corollary 3.11 then implies that

$$\begin{aligned} \mathrm{Tor}_i^{\mathcal{F}_{B\mathfrak{m}} \mathcal{M}}(R/\mathfrak{m}_{\mathfrak{m}}, C_{\mathfrak{m}}) &\cong \mathrm{Tor}_i^{\mathcal{F}_B \mathcal{M}}(R/\mathfrak{m}, C)_{\mathfrak{m}} \\ &\cong \mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(R/\mathfrak{m}, C)_{\mathfrak{m}} \\ &\cong \mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_{C\mathfrak{m}}}(R/\mathfrak{m}_{\mathfrak{m}}, C_{\mathfrak{m}}). \end{aligned}$$

Because this is so for $i = 0$ and some $i \geq 1$, Theorem 4.8 implies that $B_{\mathfrak{m}} \cong R_{\mathfrak{m}} \cong C_{\mathfrak{m}}$. This holds for all \mathfrak{m} , so Fact 1.21 implies $B \approx R \approx C$. \square

The next two results are proved similarly.

Corollary 4.15. *The following conditions are equivalent:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(X, Y) \cong \mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(Y, X)$ for all $i \geq 0$ and for all R -modules X, Y .
- (ii) $\mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(X, Y) \cong \mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(Y, X)$ for all $i \geq 0$ and for all R -modules X, Y .
- (iii) $\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(C, R/\mathfrak{m}) \cong \mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(R/\mathfrak{m}, C)$ for $i = 0$, for some $i \geq 1$, and for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)$.
- (iv) $\mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(C, R/\mathfrak{m}) \cong \mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(R/\mathfrak{m}, C)$ for $i = 0$, for some $i \geq 1$, and for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)$.
- (v) $C \approx R$.

Corollary 4.16. *The following conditions are equivalent:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(X, Y) \cong \mathrm{Tor}_i^R(X, Y)$ for all $i \geq 0$ and for all R -modules X, Y .
- (ii) $\mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(X, Y) \cong \mathrm{Tor}_i^R(X, Y)$ for all $i \geq 0$ and for all R -modules X, Y .
- (iii) $\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(C, R/\mathfrak{m}) \cong \mathrm{Tor}_i^R(C, R/\mathfrak{m})$ for some $i \geq 1$, for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)$.
- (iv) $\mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(R/\mathfrak{m}, C) \cong \mathrm{Tor}_i^R(R/\mathfrak{m}, C)$ for some $i \geq 1$, for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)$.
- (v) $C \approx R$.
- (vi) $\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(R/\mathfrak{m}, R/\mathfrak{m}) \cong \mathrm{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m})$ for some $i \geq 0$, and for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)$.
- (vii) $\mathrm{Tor}_i^{\mathcal{M} \mathcal{F}_C}(R/\mathfrak{m}, R/\mathfrak{m}) \cong \mathrm{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m})$ for some $i \geq 0$, and for all $\mathfrak{m} \in \mathfrak{m}\text{-Spec}(R)$.

Remark 4.17. In spite of the general lack of balance properties for relative Tor, one still knows, for instance, that $\mathrm{Ann}_R(\mathrm{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N)) \supseteq \mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N)$. This follows, for instance, by Theorem 3.6 since $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(\mathrm{Hom}_R(C, M))$ and $\mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(C \otimes_R N)$.

5. \mathcal{F}_C -PROJECTIVE DIMENSION AND VANISHING OF RELATIVE HOMOLOGY

In this section, C is a semidualizing R -module, and M and N are R -modules.

We begin this section with two results that are probably implicit in [15]. The first one is, in some sense, a counterpoint to Example 1.7: the example says that bounded and exact does not necessarily imply proper, while the following lemma says that bounded and proper does imply exact.

Lemma 5.1. *Assume that $\mathcal{F}_C\text{-pd}_R(M) \leq n$ and let L be a proper \mathcal{F}_C -resolution of M such that $L_i = 0$ for $i > n$. Then L^+ is exact and we have $M \in \mathcal{B}_C(R)$.*

Proof. Lemma 2.2(b) implies that the complex $\text{Hom}_R(C, L)$ is a proper flat resolution of $\text{Hom}_R(C, M)$ such that $\text{Hom}_R(C, L)_i = 0$ for $i > n$. In particular, we have $\text{fd}_R(\text{Hom}_R(C, M)) \leq n$, so $\text{Hom}_R(C, M) \in \mathcal{A}_C(R)$. By Foxby equivalence, we conclude that $M \in \mathcal{B}_C(R)$, so $M \cong C \otimes_R \text{Hom}_R(C, M)$; see Remark 1.12(a). The conditions $L \cong C \otimes_R \text{Hom}_R(C, L)$ and $M \cong C \otimes_R \text{Hom}_R(C, M)$ imply that $L^+ \cong C \otimes_R \text{Hom}_R(C, L)^+$, so Lemma 1.13(a) implies that L^+ is exact. Since each L_i is in $\mathcal{B}_C(R)$, the condition $M \in \mathcal{B}_C(R)$ follows from the two-of-three property in Remark 1.12. \square

Proposition 5.2. (a) *One has $\mathcal{F}_C\text{-pd}_R(M) \leq n$ if and only if there is an exact sequence $0 \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0$ such that each $L_i \in \mathcal{F}_C(R)$.*
 (b) $\mathcal{F}_C\text{-pd}_R(M) = \text{fd}_R(\text{Hom}_R(C, M))$.
 (c) $\mathcal{F}_C\text{-pd}_R(C \otimes_R M) = \text{fd}_R(M)$.

Proof. (a) Assume first that $\mathcal{F}_C\text{-pd}_R(M) \leq n$ and let L be a proper \mathcal{F}_C -resolution of M such that $L_i = 0$ for $i > n$. Lemma 5.1 implies that L^+ is an exact sequence of the desired form.

Conversely, assume that there is an exact sequence

$$L^+ = (0 \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0)$$

such that each $L_i \in \mathcal{F}_C(R)$. The two-of-three property for Bass classes implies that $M \in \mathcal{B}_C(R)$; see Remark 1.12. Lemma 1.14(a) implies that $\text{Hom}_R(C, L^+)$ is exact, that is, it is an augmented flat resolution of $\text{Hom}_R(C, M)$ such that $\text{Hom}_R(C, L)_i = 0$ for $i > n$. So we have $\text{fd}_R(\text{Hom}_R(C, M)) \leq n$. Lemma 1.5 implies that a truncation

$$T^+ = (0 \rightarrow K_n \rightarrow \text{Hom}_R(C, L_{n-1}) \rightarrow \cdots \rightarrow \text{Hom}_R(C, L_0) \rightarrow \text{Hom}_R(C, M) \rightarrow 0)$$

is an exact proper flat resolution of $\text{Hom}_R(C, M)$. From Lemma 2.2(a) we conclude that $U = C \otimes_R T$ is a proper \mathcal{F}_C -resolution of $C \otimes_R \text{Hom}_R(C, M) \cong M$ such that $U_i = 0$ for all $i \geq n$, so $\mathcal{F}_C\text{-pd}_R(M) \leq n$ as desired.

(b) The proof of Lemma 5.1 implies that $\mathcal{F}_C\text{-pd}_R(M) \geq \text{fd}_R(\text{Hom}_R(C, M))$. For the reverse inequality, assume that $\text{fd}_R(\text{Hom}_R(C, M)) = m < \infty$. Given a proper flat resolution F of $\text{Hom}_R(C, M)$, Lemma 1.5 implies that F has a truncation F' that is a proper flat resolution of $\text{Hom}_R(C, M)$ such that $F'_i = 0$ for all $i > m$. As in the previous paragraph, it follows that $\mathcal{F}_C\text{-pd}_R(M) \leq m = \text{fd}_R(\text{Hom}_R(C, M))$.

(c) To show that $\mathcal{F}_C\text{-pd}_R(C \otimes_R M) \geq \text{fd}_R(M)$, assume without loss of generality that $\mathcal{F}_C\text{-pd}_R(C \otimes_R M) < \infty$. It follows that $C \otimes_R M \in \mathcal{B}_C(R)$, so $M \in \mathcal{A}_C(R)$ by Foxby equivalence; see Remark 1.12(b). Part (b) implies that

$$\mathcal{F}_C\text{-pd}_R(C \otimes_R M) = \text{fd}_R(\text{Hom}_R(C, C \otimes_R M)) = \text{fd}_R(M)$$

as desired. The reverse inequality is verified similarly. \square

Corollary 5.3. *Let $R \rightarrow S$ be a flat ring homomorphism. Then there is an inequality $\mathcal{F}_C\text{-pd}_R(M) \geq \mathcal{F}_{S \otimes_R C}\text{-pd}_S(S \otimes_R M)$ with equality holding when S is faithfully flat over R .*

Proof. Given an R -module N , it is routine to show that $\text{fd}_R(N) \geq \text{fd}_S(S \otimes_R N)$ with equality holding when S is faithfully flat over R . This explains the second step

in the next display

$$\begin{aligned}
\mathcal{F}_C\text{-pd}_R(M) &= \text{fd}_R(\text{Hom}_R(C, M)) \\
&\geq \text{fd}_S(S \otimes_R \text{Hom}_R(C, M)) \\
&= \text{fd}_S(\text{Hom}_S(S \otimes_R C, S \otimes_R M)) \\
&= \mathcal{F}_{S \otimes_R C}\text{-pd}_S(S \otimes_R M).
\end{aligned}$$

The first and fourth steps are from Proposition 5.2(b), and the third step is by the isomorphism $S \otimes_R \text{Hom}_R(C, M) \cong \text{Hom}_S(S \otimes_R C, S \otimes_R M)$. When S is faithfully flat over R , we have equality in the second step, hence the desired conclusions. \square

Corollary 5.4. *Given an integer $n \geq 0$, the following conditions are equivalent:*

- (i) $\mathcal{F}_C\text{-pd}_R(M) \leq n$.
- (ii) $\mathcal{F}_{U^{-1}C}\text{-pd}_{U^{-1}R}(U^{-1}M) \leq n$ for each multiplicatively closed subset $U \subseteq R$.
- (iii) $\mathcal{F}_{C_{\mathfrak{p}}}\text{-pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq n$ for each $\mathfrak{p} \in \text{Spec}(R)$.
- (iv) $\mathcal{F}_{C_{\mathfrak{m}}}\text{-pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) \leq n$ for each $\mathfrak{m} \in \text{m-Spec}(R)$.

Proof. This follows from Proposition 5.2 like Corollary 5.3, using the local global principal for flat dimension. \square

Fact 5.5. Let E be an injective R -module. The next facts are essentially contained in [14, Lemmas 4.1 and 4.2]. See also Fact 1.15 and Proposition 5.2.

- (a) If N is C -injective, then $\text{Hom}_R(N, E)$ is C -flat. As a consequence, we have $\mathcal{F}_C\text{-pd}_R(\text{Hom}_R(N, E)) \leq \mathcal{I}_C\text{-id}_R(N)$. In particular, if $\mathcal{I}_C\text{-id}_R(N) < \infty$, then $\mathcal{F}_C\text{-pd}_R(\text{Hom}_R(N, E)) < \infty$. When E is faithfully injective, the converses of the first and third statements hold, and equality holds in the second statement.
- (b) If N is C -flat, then $\text{Hom}_R(N, E)$ is C -injective. As a consequence, we have $\mathcal{I}_C\text{-id}_R(\text{Hom}_R(N, E)) \leq \mathcal{F}_C\text{-pd}_R(N)$. Hence, if $\mathcal{F}_C\text{-pd}_R(\text{Hom}_R(N, E)) < \infty$, then $\mathcal{I}_C\text{-id}_R(N) < \infty$. When E is faithfully injective, the converses of the first and third statements hold, and equality holds in the second statement.

The next two results contain Theorem C from the introduction.

Theorem 5.6. *Given an integer $n \geq 0$, the following conditions are equivalent:*

- (i) $\text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, -) = 0$ for all $i > n$;
- (ii) $\text{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, -) = 0$; and
- (iii) $\mathcal{F}_C\text{-pd}_R(M) \leq n$.

Proof. Let E be a faithfully injective R -module, and set $(-)^{\vee} = \text{Hom}_R(-, E)$. Condition (i) is equivalent to the following, since E is faithfully injective:

- (i') $\text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, -)^{\vee} = 0$ for all $i > n$.

Since $\text{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, -) \cong \text{Tor}_i^{\mathcal{M}\mathcal{F}_C}(-, M)$, Theorem 3.6 and Proposition 3.14 imply that (i') is equivalent to the following:

- (i'') $\text{Ext}_{\mathcal{M}\mathcal{I}_C}^i(-, M^{\vee}) = 0$ for all $i > n$.

Similarly, condition (ii) is equivalent to the following:

- (ii'') $\text{Ext}_{\mathcal{M}\mathcal{I}_C}^{n+1}(-, M^{\vee}) = 0$.

Condition (iii) is equivalent to the following, by Fact 5.5(b):

- (iii'') $\mathcal{I}_C\text{-id}_R(M^{\vee}) \leq n$.

Fact 1.17 shows that the conditions (i'')–(iii'') are equivalent. Thus, the conditions (i)–(iii) from the statement of the theorem are equivalent. \square

Theorem 5.7. *Assume that M is finitely generated over R . Given an integer $n \geq 0$, the following conditions are equivalent:*

- (i) $\mathrm{Tor}_i^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) = 0$ for all $i > n$ and for each $\mathfrak{m} \in \mathrm{m-Spec}(R)$;
- (ii) $\mathrm{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) = 0$ for each $\mathfrak{m} \in \mathrm{m-Spec}(R)$;
- (iii) $\mathcal{P}_C\text{-pd}_R(M) \leq n$; and
- (iv) $\mathcal{F}_C\text{-pd}_R(M) \leq n$.

Proof. The implication (i) \implies (ii) is trivial, and (iv) \implies (i) is from Theorem 5.6.

(ii) \implies (iii) Assume that $\mathrm{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) = 0$ for each $\mathfrak{m} \in \mathrm{m-Spec}(R)$. The module $C_{\mathfrak{m}}$ is a semidualizing $R_{\mathfrak{m}}$ -module, so it is non-zero and finitely generated. Thus, we have

$$C \otimes_R R/\mathfrak{m} \cong C/\mathfrak{m}C \cong C_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}C_{\mathfrak{m}} \cong (R/\mathfrak{m})^{\beta_0(\mathfrak{m}; C)} \quad (5.7.1)$$

where $\beta_0(\mathfrak{m}; C) \neq 0$.

The second step in the next sequence is by Theorem 3.6:

$$\begin{aligned} 0 &= \mathrm{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, R/\mathfrak{m}) \\ &\cong \mathrm{Tor}_{n+1}^R(\mathrm{Hom}_R(C, M), (C \otimes_R R/\mathfrak{m})) \\ &\cong \mathrm{Tor}_{n+1}^R(\mathrm{Hom}_R(C, M), R/\mathfrak{m})^{\beta_0(\mathfrak{m}; C)}. \end{aligned}$$

The third step is by (5.7.1). Since $\beta_0(\mathfrak{m}; C) \neq 0$, we conclude that

$$\mathrm{Tor}_{n+1}^R(\mathrm{Hom}_R(C, M), R/\mathfrak{m}) = 0$$

for each \mathfrak{m} . Thus, Proposition 5.2(b) explains the first step in the next display

$$\mathcal{F}_C\text{-pd}_R(M) = \mathrm{fd}_R(\mathrm{Hom}_R(C, M)) = \mathrm{pd}_R(\mathrm{Hom}_R(C, M)) \leq n$$

and the remaining steps follow from the fact that $\mathrm{Hom}_R(C, M)$ is finitely generated.

(iii) \implies (iv) Assume that $\mathcal{P}_C\text{-pd}_R(M) \leq n$. Then Fact 1.15(a) provides an exact sequence

$$0 \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow M \rightarrow 0$$

such that each $L_i \in \mathcal{P}_C(R)$. In particular, we have $L_i \in \mathcal{F}_C(R)$, so Proposition 5.2(a) implies that $\mathcal{F}_C\text{-pd}_R(M) \leq n$. \square

Corollary 5.8. *If M is finitely generated, then $\mathcal{F}_C\text{-pd}_R(M) = \mathcal{P}_C\text{-pd}_R(M)$.*

Corollary 5.9. *Given a set $\{N_j\}_{j \in J}$ of R -modules, one has*

$$\mathcal{F}_C\text{-pd}_R(\coprod_j N_j) = \sup\{\mathcal{F}_C\text{-pd}_R(N_j) \mid j \in J\}.$$

Proof. Apply Theorem 3.6, Proposition 3.9(a), and Theorem 5.6. \square

We conclude this section with a two-of-three result for modules of finite \mathcal{F}_C -projective dimension.

Corollary 5.10. *Given an exact sequence $\mathbb{M} = (0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0)$ of R -module homomorphisms, one has*

$$\begin{aligned} \mathcal{F}_C\text{-pd}_R(M) &\leq \sup\{\mathcal{F}_C\text{-pd}_R(M'), \mathcal{F}_C\text{-pd}_R(M'')\} \\ \mathcal{F}_C\text{-pd}_R(M') &\leq \sup\{\mathcal{F}_C\text{-pd}_R(M), \mathcal{F}_C\text{-pd}_R(M'') - 1\} \\ \mathcal{F}_C\text{-pd}_R(M'') &\leq \sup\{\mathcal{F}_C\text{-pd}_R(M), \mathcal{F}_C\text{-pd}_R(M') + 1\}. \end{aligned}$$

In particular, if two of the modules in \mathbb{M} have finite \mathcal{F}_C -projective dimension, then so does the third module.

Proof. For each inequality, one can assume without loss of generality that two of the modules in the sequence have finite \mathcal{F}_C -projective dimension. In particular, these modules are in $\mathcal{B}_C(R)$, so the two-of-three property implies that all three modules are in $\mathcal{B}_C(R)$; see Remark 1.12. In particular, we have $\text{Ext}_R^1(C, M') = 0$, so the sequence $\text{Hom}_R(C, \mathbb{M})$ is exact. Lemma 2.3 implies that \mathbb{M} is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact, so Proposition 3.4(a) and Theorem 3.6 provide a long exact sequence

$$\cdots \text{Tor}_{i+1}^{\mathcal{F}_C \mathcal{M}}(M'', N) \rightarrow \text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M', N) \rightarrow \text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M, N) \rightarrow \text{Tor}_i^{\mathcal{F}_C \mathcal{M}}(M'', N) \cdots$$

for each R -module N . The desired inequalities follow by analyzing the vanishing in this sequence using Theorem 5.6. \square

6. PURE SUBMODULES

In this section, C is a semidualizing R -module, and M is an R -module.

Definition 6.1. An R -submodule $M' \subseteq M$ is *pure* if for every R -module N the induced map $N \otimes_R M' \rightarrow N \otimes_R M$ is injective. An exact sequence

$$\mathbb{M} = (0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0)$$

is *pure* if for every R -module N the sequence $N \otimes_R \mathbb{M}$ is exact.

Remark 6.2. An R -submodule $M' \subseteq M$ is pure if and only if the induced sequence $0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0$ is pure.

The next fact is from [17, Proposition 3].

Fact 6.3. Let $M' \subseteq M$ be an R -submodule, and consider the natural exact sequence $\mathbb{M} = (0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0)$. The following conditions are equivalent:

- (i) M' is a pure submodule of M .
- (ii) for each finitely presented (i.e., finitely generated) R -module N , the induced map $\text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, M'')$ is surjective.
- (iii) for each finitely presented R -module N , the sequence $\text{Hom}_R(N, \mathbb{M})$ is exact.

This fact yields our next result which applies, e.g., when L is semidualizing.

Proposition 6.4. Let $M' \subseteq M$ be a pure submodule, and let L be a finitely generated R -module. Then the submodule $\text{Hom}_R(L, M') \subseteq \text{Hom}_R(L, M)$ is pure.

Proof. Set $\mathbb{M} = (0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0)$, and let N be an R -module. Note that if N and L are finitely generated, then so is $N \otimes_R L$. Now use Fact 6.3 with Hom-tensor adjointness: $\text{Hom}_R(N, \text{Hom}_R(L, \mathbb{M})) \cong \text{Hom}_R(N \otimes_R L, \mathbb{M})$. \square

The next result generalizes [6, Lemma 9.1.4] and [10, Lemma 5.2(a)] in our setting. It is Theorem D from the introduction.

Theorem 6.5. Let $M' \subseteq M$ be a pure submodule. Then one has

$$\mathcal{F}_C\text{-pd}_R(M) \geq \sup\{\mathcal{F}_C\text{-pd}_R(M'), \mathcal{F}_C\text{-pd}_R(M/M') - 1\}.$$

Proof. Assume without loss of generality that $\mathcal{F}_C\text{-pd}_R(M) = n < \infty$. It follows that $M \in \mathcal{B}_C(R)$, and from [9, Proposition 2.4(a) and Theorem 3.1] we know that M' and $M'' := M/M'$ are in $\mathcal{B}_C(R)$. In particular, the sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \tag{6.5.1}$$

is $\text{Hom}_R(C, -)$ -exact, so it is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact by Lemma 2.3.

We prove that $\mathcal{F}_C\text{-pd}_R(M') \leq \mathcal{F}_C\text{-pd}_R(M)$. By Theorems 3.6 and 5.6, we have

$$\text{Tor}_{n+1}^R(\text{Hom}_R(C, M), C \otimes_R -) \cong \text{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M, -) = 0.$$

Let G be an arbitrary R -module and let

$$0 \rightarrow K_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow C \otimes_R G \rightarrow 0,$$

be a truncation of a projective resolution of $C \otimes_R G$. In the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{n+1} \otimes_R \text{Hom}_R(C, M') & \longrightarrow & P_n \otimes_R \text{Hom}_R(C, M') & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{n+1} \otimes_R \text{Hom}_R(C, M) & \longrightarrow & P_n \otimes_R \text{Hom}_R(C, M) & & \end{array}$$

the bottom row is exact, since $\text{Tor}_{n+1}^R(\text{Hom}_R(C, M), C \otimes_R G) = 0$. The two vertical arrows are injective, since $\text{Hom}_R(C, M') \subseteq \text{Hom}_R(C, M)$ is pure by Proposition 6.4. Hence, the top row of the diagram is exact, so we have

$$\text{Tor}_{n+1}^{\mathcal{F}_C\mathcal{M}}(M', G) \cong \text{Tor}_{n+1}^R(\text{Hom}_R(C, M'), C \otimes_R G) = 0$$

by Theorem 3.6. Since G was chosen arbitrarily, we conclude that $\mathcal{F}_C\text{-pd}_R(M') \leq n = \mathcal{F}_C\text{-pd}_R(M)$, by Theorem 5.6.

To complete the proof, we need only observe that Corollary 5.10 implies that $\mathcal{F}_C\text{-pd}_R(M'') - 1 \leq n$. \square

Example 6.6. Let M' and M'' be R -modules such that

$$\mathcal{F}_C\text{-pd}_R(M') < \mathcal{F}_C\text{-pd}_R(M'') < \infty.$$

The trivial exact sequence $0 \rightarrow M' \rightarrow M' \oplus M'' \rightarrow M'' \rightarrow 0$ is split hence pure, so

$$\mathcal{F}_C\text{-pd}_R(M' \oplus M'') = \mathcal{F}_C\text{-pd}_R(M'') > \sup\{\mathcal{F}_C\text{-pd}_R(M'), \mathcal{F}_C\text{-pd}_R(M'') - 1\}$$

by Proposition 5.9. Thus, we can have strict inequality in Theorem 6.5.

REFERENCES

1. B. J. Anderson and S. Sather-Wagstaff, *NAK for Ext and ascent of module structures*, preprint (2012), [arXiv:1201.3039](#).
2. L. L. Avramov and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85** (2002), 393–440. MR 2003g:16009
3. L. Bican, R. El Bashir, and E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), no. 4, 385–390. MR 1832549 (2002e:16002)
4. M. C. R. Butler and G. Horrocks, *Classes of extensions and resolutions*, Philos. Trans. Roy. Soc. London Ser. A **254** (1961/1962), 155–222. MR 0188267 (32 #5706)
5. S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc. No. **55** (1965), 39. MR 0178036 (31 #2294)
6. E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR 1753146 (2001h:16013)
7. A. J. Frankild, S. Sather-Wagstaff, and A. Taylor, *Relations between semidualizing complexes*, J. Commut. Algebra **1** (2009), no. 3, 393–436. MR 2524860
8. A. J. Frankild, S. Sather-Wagstaff, and R. A. Wiegand, *Ascent of module structures, vanishing of Ext, and extended modules*, Michigan Math. J. **57** (2008), 321–337, Special volume in honor of Melvin Hochster. MR 2492456
9. H. Holm and P. Jørgensen, *Cotorsion pairs induced by duality pairs*, J. Commut. Algebra **1** (2009), no. 4, 621–633. MR 2575834 (2011g:13034)
10. H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), no. 4, 781–808. MR 2413065

11. S. Sather-Wagstaff, *Semidualizing modules*, in preparation.
12. ———, *Bass numbers and semidualizing complexes*, Commutative algebra and its applications, Walter de Gruyter, Berlin, 2009, pp. 349–381. MR 2640315
13. S. Sather-Wagstaff, T. Sharif, and D. White, *Comparison of relative cohomology theories with respect to semidualizing modules*, Math. Z. **264** (2010), no. 3, 571–600. MR 2591820
14. ———, *AB-contexts and stability for Gorenstein flat modules with respect to semidualizing modules*, Algebr. Represent. Theory **14** (2011), no. 3, 403–428. MR 2785915
15. R. Takahashi and D. White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010), no. 1, 5–22. MR 2603458
16. W. V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]. MR 0498530 (58 #16637)
17. R. B. Warfield, Jr., *Purity and algebraic compactness for modules*, Pacific J. Math. **28** (1969), 699–719. MR 0242885 (39 #4212)

MARYAM SALIMI, DEPARTMENT OF MATHEMATICS, SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY, TEHRAN, IRAN

E-mail address: `maryamsalimi@ipm.ir`

SEAN SATHER-WAGSTAFF, DEPARTMENT OF MATHEMATICS, NDSU DEPT # 2750, PO Box 6050, FARGO, ND 58108-6050 USA

E-mail address: `sean.sather-wagstaff@ndsu.edu`

URL: `http://www.ndsu.edu/pubweb/~ssatherw/`

ELHAM TAVASOLI, DEPARTMENT OF MATHEMATICS, SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY, TEHRAN, IRAN

E-mail address: `elhamtavasoli@ipm.ir`

SIAMAK YASSEMI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEHRAN, AND SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES(IPM), TEHRAN- IRAN

E-mail address: `yassemi@ipm.ir`

URL: `http://math.ipm.ac.ir/yassemi/`